EMBEDDINGS OF GENERALIZED DANIELEWSKI SURFACES IN AFFINE SPACES

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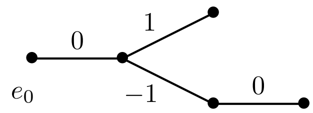
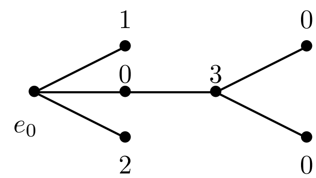
Abstract. We construct explicit embeddings of generalized Danielewski surfaces [5] in affine spaces. The equations defining these embeddings are obtain from the 2×2 minors of a matrix attached to a labelled-rooted tree $\mathfrak{g}$. The corresponding surfaces $V_{\mathfrak{g}}$ come equipped with canonical $\mathbb{C}_+$-actions which appear as the restrictions of certain $\mathbb{C}_+$-actions on the ambient affine space. We characterize those surfaces $V_{\mathfrak{g}}$ with a trivial Makar-Limanov invariant, and we complete the study of log $\mathbb{Q}$-homology planes with a trivial Makar-Limanov invariant initiated by Miyanishi and Masuda [11].

Introduction

In [5], the author introduced the notion of generalized Danielewski surface (GDS for short). This is a normal affine surface V which admits a faithfully flat morphism $q : V \to Z = \mathbb{A}^1_{\mathbb{C}}$ with reduced fibers and generic fiber $\mathbb{A}^1_{K(Z)}$, such that all but possibly one closed fiber $q^{-1}(z_0)$ are integral. For instance, a nonsingular ordinary Danielewski surface $V_{P,n} \subset \mathbb{C}[x,y,z]$ with equation $x^n z - P(y) = 0$, where P is a nonconstant polynomial with simple roots, is a GDS for the projection $pr_x : V_{P,n} \to Z = \mathrm{Spec}(\mathbb{C}[x])$. Generalized Danielewski surfaces appear naturally as locally trivial fiber bundles $\rho : V \to X$ over an affine line with a multiple origin (see e.g. [6]). In [5], the author established that every such bundle is obtain from an invertible sheaf $\mathcal{L}$ on X and a Čech 1-cocycle g with values in the dual $\mathcal{L}^\vee$ of $\mathcal{L}$. In turn, this invertible sheaf $\mathcal{L}$ and the cocycle g are encoded in a combinatorial data consisting of a rooted tree with weighted edges, which we call a *weighted rooted tree* (see [5, Theorem 4.1] and 3.3 below). Here we use rooted trees in a different way to construct embeddings of GDS's into affine spaces. More precisely, starting with certain rooted trees with weights on their nodes, which we call *labelled rooted trees*, we construct explicit ideals of certain polynomial rings. In turn, these ideals define affine surfaces which are naturally GDS's over the affine line $\mathbb{A}^1_{\mathbb{C}}$.



A weighted tree γ rooted in e_0.



A labelled tree $\mathfrak{g}$ rooted in e_0.

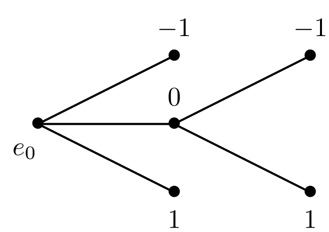
The paper is divided as follows. In section 1 we recall basic facts on rooted trees and we give a procedure to construct an affine scheme $V_{\mathfrak{g}}$ from the data consisting of a labelled rooted tree $\mathfrak{g} = (\Gamma, lb)$. These schemes $V_{\mathfrak{g}} = \mathrm{Spec}\,(B_{\mathfrak{g}})$ are naturally schemes over an affine line $Z = \mathrm{Spec}\,(\mathbb{C}\,[h])$, and they come embedded into affine spaces $\mathbb{A}_Z^{d(\Gamma)+1} = \mathrm{Spec}\,(\mathbb{C}\,[h]\,[\Gamma])$ depending on the tree Γ. In section 2, we prove the following result (theorem 2.1)

Theorem 0.1. *For every fine-labelled rooted tree $\mathfrak{g}$ (see definition 1.3 below), the scheme $V_{\mathfrak{g}} = Spec\,(B_{\mathfrak{g}})$ is a GDS over $Z = Spec\,(\mathbb{C}\,[h])$.*

For instance, the Bandman and Makar-Limanov surface [1] $V \subset \mathbb{C}\,[x, y, z, u]$ with equation

$$xz = y\left(y^2 - 1\right), \quad yu = z\left(z^2 - 1\right), \quad xu = \left(y^2 - 1\right)\left(z^2 - 1\right)$$

is a GDS over $Z = \mathrm{Spec}\,(\mathbb{C}\,[x])$ for the morphism $pr_x : V \to Z$. This data corresponds to the fine-labelled rooted tree

$\mathfrak{g} =$ 

In section 2, we recall [5] how a weighted rooted tree γ defines a GDS $q_\gamma : W_\gamma \to Z$. Then, starting with such a weighted rooted tree $\gamma = (\Gamma, w)$, we construct a fine-labelled rooted tree $\mathfrak{g}_w = (\Gamma, lb_w)$ with the same underlying tree Γ and a closed embedding $i_\gamma : W_\gamma \hookrightarrow \mathbb{A}_Z^{d(\Gamma)+1}$ inducing an isomorphism $W_\gamma \simeq V_{\mathfrak{g}_w}$. This leads to the following result (theorem 3.1)

Theorem 0.2. *Every GDS $q : V \to Z$ is Z-isomorphic to a GDS $q_{\mathfrak{g}} : V_{\mathfrak{g}} \to Z$ for an appropriate fine-labelled rooted tree $\mathfrak{g}$.*

In section 3 we study $\mathbb{G}_{a,Z}$-actions on the GDS's $q_{\mathfrak{g}} : V_{\mathfrak{g}} = \mathrm{Spec}\,(B_{\mathfrak{g}}) \to Z$. We construct explicit locally nilpotent derivations of the algebras $B_{\mathfrak{g}}$. In [8, 9], Makar-Limanov proved that every $\mathbb{C}_+$-action on an ordinary Danielewski surface $V_{P,n} \subset \mathrm{Spec}\,(\mathbb{C}\,[x, y, z])$ is the restriction of a $\mathbb{C}_+$-action on $\mathbb{A}^3_{\mathbb{C}} = \mathrm{Spec}\,(\mathbb{C}\,[x, y, z])$. We prove the following general result (theorem 4.9).

Theorem 0.3. *Every $\mathbb{G}_{a,Z}$-action on a GDS $q : V_{\mathfrak{g}} \to Z$ defined by a fine-labelled rooted tree $\mathfrak{g} = (\Gamma, lb)$ is induced by a $\mathbb{G}_{a,Z}$-action on the ambient space $\mathbb{A}_Z^{d(\Gamma)+1} = Spec\,(\mathbb{C}\,[h]\,[\Gamma])$.*

In section 4, we consider GDS's $q_{\mathfrak{g}}:V_{\mathfrak{g}}\rightarrow Z$ with a trivial Makar-Limanov invariant. As a consequence of [5, Theorem 7.2], we obtain the following criterion (theorem 5.1).

Theorem 0.4. *A GDS $q_{\mathfrak{g}}:V_{\mathfrak{g}}\rightarrow Z$ has a trivial Makar-Limanov invariant if and only if $\mathfrak{g}$ is a fine-labelled comb (see 5.1 below).*

This leads to the following description (see 5.2 below). Let $P_1,\ldots,P_n\in\mathbb{C}\left[T\right]$ be a collection of nonconstant polynomials with simple roots, one of these root, say $\lambda_{i,1}$, $1\leq i\leq n$, being distinguished. We let

$$P_i\left(T\right)=\prod_{j=1}^{r_i}\left(T-\lambda_{i,j}\right)=\left(T-\lambda_{i,1}\right)\tilde{P}_i\left(T\right)\in\mathbb{C}\left[T\right],\quad 1\leq i\leq n.$$

Then every GDS with a trivial Makar-Limanov is isomorphic, for an appropriate choice of the polynomials $P_1,\ldots,P_n$, to a surface

$$V_{P_1,\ldots,P_n}\subset\mathbb{A}^{n+2}_{\mathbb{C}}=\mathrm{Spec}\left(\mathbb{C}\left[x\right]\left[y_1,\ldots,y_{n+1}\right]\right)$$

with equations

$$\left\{\begin{array}{lcll} xy_{i+1} & = & \left(\displaystyle\prod_{k=1}^{i-1}\tilde{P}_k\left(y_k\right)\right)P_i\left(y_i\right) & 1\leq i\leq n \\ \left(y_{j-1}-\lambda_{j-1,1}\right)y_{i+1} & = & y_j\left(\displaystyle\prod_{k=j}^{i-1}\tilde{P}_k\left(y_k\right)\right)P_i\left(y_i\right) & 2\leq j\leq i\leq n \end{array}\right.,$$

where, by convention, $\displaystyle\prod_{k=j}^{i-1}\tilde{P}_k\left(y_k\right)=1$ if $i=j$. From this description, we recover the following characterization of nonsingular ordinary Danielewski surfaces, due to Bandman and Makar-Limanov [1] (5.4).

Theorem 0.5. *For a GDS $q:V\rightarrow Z$ with a trivial Makar-Limanov invariant, the following are equivalent.*

1) V admits a free $\mathbb{G}_{a,Z}$-action.

2) The canonical sheaf ω_V of V is trivial.

3) V is isomorphic to an ordinary Danielewski surface $V_{P,1}\subset\mathbb{A}^3_{\mathbb{C}}=Spec\left(\mathbb{C}\left[x,y,z\right]\right)$ with the equation $xz-P\left(y\right)=0$ for a certain nonconstant polynomial P with simple roots.

Following a remark of the author, Miyanishi and Masuda [11] proved that every nonsingular $\mathbb{Q}$-homology plane with a trivial Makar-Limanov invariant is a cyclic quotient of an ordinary Danielewski surface $V\subset\mathrm{Spec}\left(\mathbb{C}\left[x,t,z\right]\right)$ with equation $xz=t^m-1$ by a $\mathbb{Z}_m$-action $\left(x,t,z\right)\mapsto\left(\varepsilon x,\varepsilon^q y,\varepsilon^{-1}z\right)$, where ε is a primitive m-the root of unity and $\gcd\left(q,m\right)=1$. More generally, the author proved in [5, theorem 7.7] that every normal affine surface S with a trivial Makar-Limanov invariant is a cyclic quotient of a GDS V. In case that S is a log $\mathbb{Q}$-homology plane, this GDS can be determined explicitly. This leads to the following result (theorem 5.6), see also [2].

Theorem 0.6. *Every log $\mathbb{Q}$-homology plane $S\not\simeq\mathbb{A}^2_{\mathbb{C}}$ with a trivial Makar-Limanov invariant is isomorphic to a quotient of an ordinary Danielewski surface $V\subset Spec\left(\mathbb{C}\left[x,t,z\right]\right)$ with equation $xz=t^n-1$ by a $\mathbb{Z}_m$-action $\left(x,t,z\right)\mapsto\left(\varepsilon x,\varepsilon^q t,\varepsilon^{-1}z\right)$, where ε is a primitive m-th root of unity, n divides m and $\gcd\left(q,m/n\right)=1$.*

1. Affine schemes defined from labelled rooted trees

In this section we explain how to construct affine schemes from labelled rooted trees.

Basics on rooted trees.
Let $G=(N,\leq)$ be a nonempty, finite, partially ordered set (a *poset*, for short). The elements of N are sometimes called the *nodes* of G. A totally ordered subset $N'\subset N$ is called a *chain of length* $l(N')=\mathrm{Card}(N')-1$. *A* chain which is maximal for the inclusion is called a *maximal chain.* For every $e\in N$, we let

$$(\uparrow e)_G=\left\{e'\in N, e'\geq e\right\} \quad \text{and} \quad (\downarrow e)_G=\left\{e'\in N, e'\leq e\right\}$$

For every $e<e'$, $e,e'\in N$, we let $[e',e]_G:=(\uparrow e')_G\cap(\downarrow e)_G$. A pair $e<e'$ such that $[e',e]_G=\{e'<e\}$ is called an edge of G, and we denote the set of all edges in G by $E(G)$.

Definition 1.1. A *rooted tree* Γ is poset $\Gamma=(N(\Gamma),\leq)$ with a unique minimal element e_0 called the *root*, and such that $(\downarrow e)_\Gamma$ is a chain for every $e\in N(\Gamma)$.

1.2. The maximal elements of a rooted tree $\Gamma=(N(\Gamma),\leq)$ are called the *leaves* of Γ. We denote the set of those elements by $\mathrm{Leaves}(\Gamma)$. An element of $N(\Gamma)\setminus\mathrm{Leaves}(\Gamma)$ is called a *parent*, and we denote the set of those nodes by $\mathbf{P}(\Gamma)$. Given $e\in N(\Gamma)\setminus\{e_0\}$, an element of the chain $\mathrm{Anc}(e)=(\downarrow e)\setminus\{e\}$ is called an *ancestor* of e. The *parent of* e is the maximal element $\mathrm{Par}(e)$ of $\mathrm{Anc}(e)$. More generally, the n-th *ancestor* of e is defined recursively by $\mathrm{Par}^n(e)=\mathrm{Par}\left(\mathrm{Par}^{n-1}(e)\right)\in\mathrm{Anc}(e)$. Given two different nodes $g,g'\in N(\Gamma)$, the *first common ancestor* of g *and* g' is the maximal element $\mathrm{Anc}(g,g')$ of the chain $\mathrm{Anc}(g)\cap\mathrm{Anc}(g)$. Given $e\in\mathbf{P}(\Gamma)$, the minimal elements of $(\uparrow e)_\Gamma\setminus\{e\}$ are called the *children* of e, and we denote the set of those nodes by $\mathrm{Child}_\Gamma(e)$. The *degree* $\deg_\Gamma(e)$ of a node e is the number of its children. Given $e\in N(\Gamma)$, the *maximal rooted subtree of* Γ *rooted in* e is the tree $\Gamma(e)=((\uparrow e)_\Gamma,\leq)$. A node $e\in N(\Gamma)$ such that $l((\downarrow e))=n$ is said to be at *level* n, and we denote the set of those nodes by $N_n(\Gamma)$. The maximal chains of a rooted tree Γ are the chains

$$(\downarrow e)_\Gamma=\left\{e_0<e_1<\cdots<e_{n_f-1}<e_{n_f}=e\right\},\quad e\in\mathrm{Leaves}(\Gamma). \tag{1.1}$$

The *height* $h(\Gamma)$ of Γ is the maximum of the lengths $l((\downarrow e)_\Gamma)=n_e$, $e\in\mathrm{Leaves}(\Gamma)$.

In 1.3 and 1.4 below, we introduce two different notions of weights on a rooted tree Γ.

Definition 1.3. A *labelling* (with values in $\mathbb{C}$) on a tree Γ rooted in e_0 is a function

$$lb: N(\Gamma)\setminus\{e_0\}\to\mathbb{C}$$

A labelling lb is called *fine* if $lb(f)\neq lb(f')$ whenever f and f' share the same parent $e\in\mathbf{P}(\Gamma)$. A rooted tree Γ equipped with a labelling lb will be referred to as a *labelled rooted tree* and will be denoted by $\mathfrak{g}=(\Gamma,lb)$. A rooted tree Γ equipped with a fine labelling lb will be referred to as a *fine-labelled rooted tree.* For every $e\in\mathbf{P}(\Gamma)$, we let $\mathfrak{g}(e)=(\Gamma(e),lb)$ be the maximal subtree of Γ rooted in e, equipped with the restriction of the labelling lb to $N(\Gamma(e))\setminus\{e\}$.

Definition 1.4. A *weight function* (with values in $\mathbb{C}$) on a rooted tree Γ is a function

$$w: E(\Gamma)\to\mathbb{C},$$

i.e. a function which assign a complex number $w([e,f]_\Gamma)$ to every edge $[e,f]_\Gamma$ of Γ, such that $w([e,f])\neq w([e,f'])$ whenever f and f' share the same parent $e\in\mathbf{P}(\Gamma)$. A rooted tree Γ equipped with a weight function w will be referred to as a *weighted rooted tree* and will be denoted by $\gamma=(\Gamma,w)$. For every $e\in\mathbf{P}(\Gamma)$, we let $\gamma(e)=(\Gamma(e),w)$ be the maximal subtree of Γ rooted in e, equipped with the restriction of the weight function w to $E(\Gamma(e))$.

Affine scheme associated to a labelled rooted tree.
In this subsection we give a general procedure for constructing an affine scheme $V_{\mathfrak{g}}$ from the data consisting of a labelled rooted tree $\mathfrak{g} = (\Gamma, lb)$. To fix the notation, we let $A = \mathbb{C}[h]$ be a polynomial ring in one variable, $z_0 = (h) \in Z = \operatorname{Spec}(A)$ and $Z_* = Z \setminus \{z_0\} \simeq \operatorname{Spec}(A_h)$. We let $pr_Z : \mathbb{A}^1_Z = \operatorname{Spec}(A[X_0]) \to Z$ be the trivial line bundle over Z.

Definition 1.5. Given a rooted tree Γ, we let $\mathbf{S}(M_\Gamma)$ be the symmetric algebra of the free A-module M_Γ with basis $(X_e)_{e\in\mathbf{P}(\Gamma)}$. This is a polynomial ring over A in

$$d(\Gamma) := \sum_{i=1}^{h(\Gamma)} \operatorname{Card}(N_{i-1}(\Gamma) \setminus \operatorname{Leaves}(\Gamma)) \tag{1.2}$$

variables. Then we let $A[\Gamma] = A[X_0] \otimes_A \mathbf{S}(M_\Gamma)$.

Notation 1.6. If $e' \in \mathbf{P}(\Gamma)$ is the parent of a given $e \in \mathbf{P}(\Gamma) \setminus \{e_0\}$ then we will sometimes denote $X_{e'} \in A[\Gamma]$ as $X_{\operatorname{Par}(e)}$. We also extend this relationship between the variables X_e, $e \in \mathbf{P}(\Gamma)$, by letting $X_{\operatorname{Par}(e_0)} = X_0 \in A[\Gamma]$.

For every node $e \in \mathbf{P}(\Gamma)$ of a given labelled rooted tree $\mathfrak{g} = (\Gamma, lb)$, we introduce, in 1.7-1.9 below, three polynomials $S_e(\mathfrak{g}), R_e(\mathfrak{g}), Q_e(\mathfrak{g}) \in A[\Gamma]$, defined recursively through the labelling lb.

Definition 1.7. For every $e \in \mathbf{P}(\Gamma)$ and every subset $J \subset \operatorname{Child}(e)$ we let

$$S_e^J(\mathfrak{g}) = \prod_{e'' \in (\operatorname{Child}(e) \setminus J)} \left(X_{\operatorname{Par}(e)} - lb(e'')\right) \in \mathbb{C}\left[X_{\operatorname{Par}(e)}\right] \subset A[\Gamma].$$

We call $S_e(\mathfrak{g}) := S_e^{\emptyset}(\mathfrak{g})$ *the sibling polynomial* of e. This is a polynomial of degree $\deg_\Gamma(e)$. The roots of $S_e(\mathfrak{g})$ are simple if and only if $lb : N(\Gamma) \setminus \{e_0\} \to \mathbb{C}$ restricts to a fine labelling on the subtree $(\{e\} \cup \operatorname{Child}(e), \leq_\Gamma)$ of Γ.

Definition 1.8. The *root polynomial* of $e \in N(\Gamma)$ is the polynomial $R_e(\mathfrak{g})$ defined recursively by $R_{e_0}(\mathfrak{g}) = 1$ and

$$R_e(\mathfrak{g}) = S_{\operatorname{Par}(e)}^{\{e\}}(\mathfrak{g})\, R_{\operatorname{Par}(e)}(\mathfrak{g}) \in \mathbb{C}\left[X_0, (X_{e'})_{e' \in \operatorname{Anc}(\operatorname{Par}(e))}\right] \subset A[\Gamma].$$

Definition 1.9. For every $e \in \mathbf{P}(\Gamma)$, we let

$$Q_e(\mathfrak{g}) \quad = \quad S_e(\mathfrak{g})\, R_e(\mathfrak{g}) \in \mathbb{C}\left[X_0, (X_{e'})_{e' \in \operatorname{Anc}(e)}\right] \subset A[\Gamma].$$

Definition 1.10. Given a labelled rooted tree $\mathfrak{g} = (\Gamma, lb)$, we let

$$M(\mathfrak{g}) = \left(M_0, (M_e)_{e \in \mathbf{P}(\Gamma)}\right) \in \operatorname{Mat}_{2,(\operatorname{Card}(\mathbf{P}(\Gamma))+1)}(A[\Gamma])$$

be the matrix with the columns

$$M_0 = \begin{pmatrix} h \\ 1 \end{pmatrix} \in \operatorname{Mat}_{2,1}(A[\Gamma]) \quad \text{and} \quad M_e = \begin{pmatrix} Q_e(\mathfrak{g}) \\ X_e \end{pmatrix} \in \operatorname{Mat}_{2,1}(A[\Gamma]), \quad e \in \mathbf{P}(\Gamma).$$

Definition 1.11. Given a labelled rooted tree $\mathfrak{g} = (\Gamma, lb)$, we let $I_{\mathfrak{g}} \subset A[\Gamma]$ be the ideal generated by the polynomials

$$\left\{ \begin{array}{lcll} \Delta_{0,e}(\mathfrak{g}) & = & \det(M_0, M_e) & e \in \mathbf{P}(\Gamma) \\ \Delta_{e',e}(\mathfrak{g}) & = & R_{e''}^{-1}(\mathfrak{g}) \det(M_{e'}, M_e) & \begin{array}{c} (e, e') \in \mathbf{P}(\Gamma) \times \operatorname{Anc}(e) \\ e'' = \operatorname{Child}(e') \cap (\downarrow e) \end{array} \end{array} \right. . \tag{1.3}$$

The *affine scheme* $V_{\mathfrak{g}}$ *associated to* $\mathfrak{g}$ is the closed subscheme

$$V_{\mathfrak{g}} = \mathrm{Spec}\,(B_{\mathfrak{g}}) \subset \mathrm{Spec}\,(A\,[\Gamma])\,,\ \text{with}\ B_{\mathfrak{g}} = A\,[\Gamma]\,/I_{\mathfrak{g}}$$

Remark 1.12. For every pair $(e,e') \in \mathbf{P}(\Gamma) \times \mathrm{Anc}\,(e)$,

$$\Delta_{e',e}(\mathfrak{g}) \;=\; \left(X_{\mathrm{Par}(e')} - lb\left(e''\right)\right) X_e - X_{e'} Q_{e',e}(\mathfrak{g})\,, \tag{1.4}$$

where $e'' = \mathrm{Child}\,(e') \cap (\downarrow e)$ and $Q_{e',e}(\mathfrak{g}) = R_{f'}^{-1}(\mathfrak{g})\, Q_e(\mathfrak{g}) \in \mathbb{C}\left[(X_g)_{g \in [e', \mathrm{Par}(e)]_{\Gamma}}\right]$.

Remark 1.13. For every pair $(e,e') \in \mathbf{P}(\Gamma) \times \mathrm{Anc}\,(e)$ as above, the polynomials $\Delta_{0,e}(\mathfrak{g})$, $\Delta_{0,e'}(\mathfrak{g})$ and $\Delta_{e',e}(\mathfrak{g})$ satisfy the syzygy relation

$$h\Delta_{e',e}(\mathfrak{g}) \;=\; \left(X_{\mathrm{Par}(e')} - lb\left(e''\right)\right) \Delta_{0,e}(\mathfrak{g}) - Q_{e',e}(\mathfrak{g})\, \Delta_{0,e'}(\mathfrak{g})\,. \tag{1.5}$$

Remark 1.14. Given a pair $(g_1,g_2) \in \mathbf{P}(\Gamma) \times \mathbf{P}(\Gamma)$ such that $g_i \notin (\downarrow g_j)$, $i,j \in \{1,2\}$, we let $e \in \mathbf{P}(\Gamma)$ be the first common ancestor of g_1 and g_2 by e, and we let $e_i = \mathrm{Child}\,(e) \cap (\downarrow g_i)$, $i = 1,2$. Then

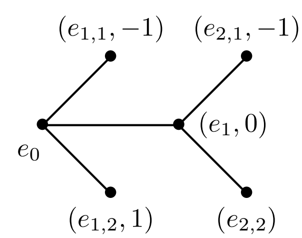
$$\Delta_{g_1,g_2} \;:=\; \left(R_e(\mathfrak{g})\, S_e^{\{e_1,e_2\}}(\mathfrak{g})\right)^{-1} \det\left(M_{g_1}, M_{g_2}\right) \tag{1.6}$$

$$\;=\; Q_{e,g_1}(\mathfrak{g})\, \Delta_{e,g_2}(\mathfrak{g}) - Q_{e,g_2}(\mathfrak{g})\, \Delta_{e,g_1}(\mathfrak{g}) \tag{1.7}$$

This means that $I_{\mathfrak{g}}$ coincides with the ideal generated by the *simplified* 2×2 *minors* of $M(\mathfrak{g})$, i.e. the 2×2 minors of $M(\mathfrak{g})$ simplified by a common factor according to the rules (1.3) and (1.6).

Notation 1.15. Hereafter, we will usually write P instead of $P(\mathfrak{g})$, where $P(\mathfrak{g})$ stands for any of the polynomials $S_e(\mathfrak{g})$, $R_e(\mathfrak{g})$, $Q_e(\mathfrak{g})$, $\Delta_{0,e}(\mathfrak{g})$ and $\Delta_{e',e}(\mathfrak{g})$ attached to a labelled rooted tree $\mathfrak{g}$.

Example 1.16. Consider the following fine-labelled rooted tree

$\mathfrak{g} =$ 

The corresponding algebra is $A\,[\Gamma] = \mathbb{C}\,[h]\,[X_0, X_{e_0}, X_{e_1}]$, and the associated matrix is

$$M(\mathfrak{g}) = \begin{pmatrix} h & X_0\left(X_0^2-1\right) & \left(X_0^2-1\right)\left(X_{e_0}^2-1\right) \\ 1 & X_{e_0} & X_{e_1} \end{pmatrix}.$$

Therefore $V_{\mathfrak{g}}$ is the Bandman and Makar-Limanov surface [1] with the equations

$$hX_{e_0} = X_0\left(X_0^2-1\right), \quad X_0X_{e_1} = X_{e_0}\left(X_{e_0}^2-1\right), \quad hX_{e_1} = \left(X_0^2-1\right)\left(X_{e_0}^2-1\right).$$

This is a *GDS* over $Z = \mathrm{Spec}\,(A)$ for the morphism $q_{\mathfrak{g}} : V_{\mathfrak{g}} \to Z$ induced by the inclusion $A \hookrightarrow B_{\mathfrak{g}}$. This $\mathbb{A}^1$-fibration $q_{\mathfrak{g}} : V_{\mathfrak{g}} \to Z$ restricts to the trivial line bundle over $Z_* = Z \setminus \{z_0\}$, whereas $q_{\mathfrak{g}_1}^{-1}(z_0)$ is the disjoint union of 4 curves

$$C_{e_{1,1}}, C_{e_{1,2}} \simeq \mathrm{Spec}\,(\mathbb{C}\,[X_{e_0}]) \quad \text{and} \quad C_{e_{2,1}}, C_{e_{2,2}} \simeq \mathrm{Spec}\,(\mathbb{C}\,[X_{e_1}])\,.$$

2. *GDS*'s defined by a fine-labelled rooted trees

This section is devoted to the proof of the following result (see also theorem 3.1 below).

Theorem 2.1. *For every* fine-labelled *rooted tree* $\mathfrak{g} = (\Gamma, lb)$*, the scheme* $V_{\mathfrak{g}} = Spec\,(B_{\mathfrak{g}})$ *is a* *GDS over* $Z = Spec\,(A)$ *for the morphism* $q_{\mathfrak{g}} : V_{\mathfrak{g}} \to Z$ *induced by the inclusion* $A \hookrightarrow B_{\mathfrak{g}}$*.*

Example 2.2. Consider the ordinary Danielewski surface $V = V_{P,1} \subset \mathrm{Spec}\,(\mathbb{C}\,[x, y, z])$ with equation $xz = P\,(y)$, where P is a polynomial with $r \geq 1$ simple roots $y_1, \ldots, y_r$. It is a *GDS* via $pr_x : V \to Z = \mathrm{Spec}\,(\mathbb{C}\,[x])$. Since $xz - P\,(y) \in \mathbb{C}\,[x, y, z]$ is an irreducible polynomial, V is an irreducible. By the Jacobian criterion, it is also nonsingular. Thus pr_x is a flat morphism. It restricts to a trivial line bundle over $Z_* = \mathrm{Spec}\left(\mathbb{C}\left[x, x^{-1}\right]\right)$. Since the roots of P are simple, the polynomials $y - y_i \in \mathbb{C}\,[y]$, $1 \leq i \leq r$ are relatively prime and so,

$$pr_x^{-1}\,(0) = \mathrm{Spec}\,(\mathbb{C}\,[x, y, z]\,/\,(x, xz - P\,(y))) \simeq \bigsqcup_{i=1}^{r} \mathrm{Spec}\,(\mathbb{C}\,[y, z]\,/\,(y - y_i))$$

is the disjoint union of curves $C_i \simeq \mathrm{Spec}\,(\mathbb{C}\,[z])$, $1 \leq i \leq r$. In general, a scheme $V_{\mathfrak{g}}$ associated to a fine-labelled rooted tree $\mathfrak{g}$ is given by more than one equation. Therefore, it is hard to check directly if it is irreducible or not. In particular, a necessary condition for $q_{\mathfrak{g}} : V_{\mathfrak{g}} \to Z$ to be flat is that no irreducible or embedded component of $V_{\mathfrak{g}}$ is supported on the fiber $q_{\mathfrak{g}}^{-1}\,(z_0)$. Let us give a direct proof of the flatness of $pr_x : V \to Z = \mathrm{Spec}\,(\mathbb{C}\,[x])$ which illustrates the constructions used below. Since pr_x restricts to a trivial line bundle over Z_*, it suffices to show that $\mathcal{O}_{V,v}$ is a torsion free $\mathcal{O}_{Z,z_0}$-module for every point $v \in V$ such that $pr_x\,(v) = z_0 = Z \setminus Z_*$. For every $i \in I$, we let

$$R_i\,(y) = \prod_{j \neq i} (y - y_j) = (y - y_i)^{-1}\, P\,(y) \in \mathbb{C}\,[x, y, z]\,.$$

Since the roots of $P\,(y)$ are simple, the principal open subsets

$$U_i = V \cap D\,(R_i) \;\simeq\; \mathrm{Spec}\,(\mathbb{C}\,[x, y, z]\,[t]\,/\,(xz - P\,(y)\,, R_i\,(y)\,t - 1)) \;\simeq\; \mathrm{Spec}\,(\mathbb{C}\,[x, z, t]\,/\,(tR_i\,(y_i + xzt) - 1))\,, \quad 1 \leq i \leq r.$$

cover V. Since x does not divide $tR_i\,(y_i + xzt) - 1$, U_i is flat over Z. Thus V is covered by open subsets U_i, $1 \leq i \leq r$, flat over Z, whence is flat over Z too. We also deduce directly from this description that no irreducible component of V is supported on the fiber $pr_x^{-1}\,(0)$. Indeed, the subscheme $V\,(R_i) \cap V$ is the union of the closures of the curves

$$S_j = \{y = y_j, z = 0, x \neq 0\} \cap V \subset V \setminus pr_x^{-1}\,(0)\,, \quad j \neq i,$$

and of the irreducible components C_j, $j \neq i$, of the fiber $pr_x^{-1}\,(0)$. This shows that for for every $1 \leq i \leq r$ the generic point of $(C_i)_{red}$ is contained in the closure in V of the open subset $V \setminus pr_x^{-1}\,(0)$.

The proof divides as follows. In subsection 1 we show that $q_{\mathfrak{g}} : V_{\mathfrak{g}} \to Z$ is a flat morphism. Then is subsection 2 we describe the fiber $q_{\mathfrak{g}}^{-1}\,(z_0)$.

1. Flatness of $q_{\mathfrak{g}} : V_{\mathfrak{g}} \to Z$.

In this subsection we prove the following result.

Proposition 2.3. *If* $\mathfrak{g}$ *is a fine-labelled rooted tree then* $q_{\mathfrak{g}} : V_{\mathfrak{g}} \to Z$ *is a flat morphism restricting over* Z_* *to a trivial line bundle.*

2.4. If $N(\Gamma)=\{e_0\}$ then $B_{\mathfrak{g}}=A[X_0]$ and $q_{\mathfrak{g}}=pr_Z$ is a flat morphism. Otherwise, (1.5) implies that the image of $I_{\mathfrak{g}}$ in $A_h[\Gamma]=A[\Gamma]\otimes_A A_h$ is generated by the polynomials

$$\delta_{0,e}=h^{-1}\Delta_{0,e}=X_e-h^{-1}Q_e,\quad e\in\mathbf{P}(\Gamma).\tag{2.1}$$

Since Q_e only involves the variables X_0 and $X_{e'}$, $e'\in\mathrm{Anc}(e)$, we recursively arrive at an A_h-algebras isomorphism $A_h[\Gamma]/I_0\simeq A_h[X_0]$. Thus $q_{\mathfrak{g}}$ restricts to a trivial line bundle over $Z_*\simeq\mathrm{Spec}(A_h)$ and so, $\mathcal{O}_{V_{\mathfrak{g}},x}$ is a flat $\mathcal{O}_{Z,z}$-module for every $x\in V_{\mathfrak{g}}$ such that $q_{\mathfrak{g}}(x)=z\in Z_*$. Given $x\in V_{\mathfrak{g}}$ such that $q_{\mathfrak{g}}(x)=z_0$, $\mathcal{O}_{V_{\mathfrak{g}},x}$ is a flat $\mathcal{O}_{Z,z_0}$-module provided that h is not a zero divisor in $\mathcal{O}_{V_{\mathfrak{g}},x}$. So it suffices to show that h is not a zero divisor in $B_{\mathfrak{g}}$. In turn, it suffices to find $g_1,\ldots,g_r\in B_{\mathfrak{g}}$ which generate the unit ideal, such that h is not a zero divisor in $(B_{\mathfrak{g}})_{g_i}$ for every $1\leq i\leq r$. The following lemma says that $V_{\mathfrak{g}}$ admits a natural covering by principal open subsets.

Lemma 2.5. *The scheme $V_{\mathfrak{g}}$ is covered by the principal open subsets*

$$U_e=D(R_e)\cap V_{\mathfrak{g}}\simeq \mathit{Spec}\left(A[\Gamma][T]/(I_{\mathfrak{g}},R_eT-1)\right),\quad e\in \mathit{Leaves}(\Gamma).$$

Proof. Let us show that the ideal $J\subset A[\Gamma]$ generated by the polynomials R_e, $e\in\mathrm{Leaves}(\Gamma)$, is the unit ideal of $A[\Gamma]$. Given $e'\in\mathbf{P}(\Gamma)$ such that $\mathrm{Child}(e')\subset\mathrm{Leaves}(\Gamma)$, the polynomials $S_{e'}^{\{e\}}\in\mathbb{C}\left[X_{\mathrm{Par}(e')}\right]$, $e\in\mathrm{Child}(e')$, are relatively prime as $\mathfrak{g}$ is fine-labelled. Thus there exist polynomials $\Lambda_e\in A[\Gamma]$, $e\in\mathrm{Child}(e')$, such that

$$R_{e'}=R_{e'}\sum_{e\in\mathrm{Child}(e')}\Lambda_e S_{e'}^{\{e\}}=\sum_{e\in\mathrm{Child}(e')}\Lambda_e R_e$$

and so, $R_{e'}\in J$. By repeatedly applying the same argument, we conclude that $1=R_{e_0}\in J$. Therefore the open subsets $D(R_e)$, $e\in\mathrm{Leaves}(\Gamma)$, cover $\mathrm{Spec}(A[\Gamma])$ and so, the open subsets $(U_e)_{e\in\mathrm{Leaves}(\Gamma)}$ cover $V_{\mathfrak{g}}$. □

Given $e\in\mathrm{Leaves}(\Gamma)$, we have an isomorphism of $\mathbb{C}$-algebras

$$\Gamma\left(U_e,\mathcal{O}_{V_{\mathfrak{g}}}\right)\simeq(B_{\mathfrak{g}})_{\overline{R_e}}\simeq A[\Gamma][T]/(I_{\mathfrak{g}},G_e),$$

where $G_e=TR_e-1$ and $\overline{R_e}$ denotes the image of $R_e\in A[\Gamma]$ in $B_{\mathfrak{g}}$. In view of 2.4 above, the following proposition completes the proof of proposition 2.3.

Proposition 2.6. *There exists a nonzero polynomial $P_e\in A\left[X_{Par(e)},T\right]$ such that*

$$(B_{\mathfrak{g}})_{\overline{R_e}}\simeq A\left[X_{Par(e)},T\right]/(TP_e-1).$$

In particular, h is not a zero divisor in $(B_{\mathfrak{g}})_{\overline{R_e}}$ as it does not divide (TP_e-1).

Proof. With the notation of (1.1), we let

$$L_i=R_{e_{i+1}}^{-1}R_e\in\mathbb{C}\left[X_{e_i},\ldots,X_{e_{n_e-1}}\right]\subset A[\Gamma],\quad 0\leq i\leq n_e-2.$$

Given $e'\in\mathbf{P}(\Gamma)\setminus(\downarrow e)$, the first common ancestor $\mathrm{Anc}(e',e)$ of e and e' is a node e_i for a certain indice $i\leq n_e-1$, and $e''=\mathrm{Child}(e_i)\cap(\downarrow e)\neq e_{i+1}$. We let

$$K_{e'}=\left(X_{e_{i-1}}-lb\left(e''\right)\right)^{-1}R_e\in\mathbb{C}\left[X_0,X_{e_0},\ldots,X_{e_{n_e-1}}\right]\subset A[\Gamma].$$

Modulo the ideal I_0 generated by the polynomials

$$\left\{\begin{array}{lllr} \delta_{0,i} & = & -\left(X_{e_{i-1}}-lb\left(e_{i+1}\right)\right)+hX_{e_i}L_iT & 0\leq i\leq n_e-2 \\ & = & TL_i\Delta_{0,e_i}+\left(X_{e_{i-1}}-lb\left(e_{i+1}\right)\right)G_e & \\ & & & \\ \delta_{e'} & = & X_{e'}-TK_{e'}X_{e_i}Q_{e_i,e'} & e'\in\mathbf{P}\left(\Gamma\right)\setminus\left(\downarrow e\right) \\ & = & TK_{e'}\Delta_{e_i,e'}-X_{e'}G_e & \mathrm{Anc}\left(e',e\right)=e_i \end{array}\right.$$

we can recursively eliminate all the variables in $A\left[\Gamma\right]\left[T\right]$ but $X_{\mathrm{Par}(e)}$ and T. Therefore, there exists a nonzero polynomial $P_e\in A\left[X_{\mathrm{Par}(e)},T\right]$ such that $R_e\equiv P_e$ mod I_0 and so,

$$A\left[\Gamma\right]\left[T\right]/\left(I_0,G_e\right)\ \simeq\ A\left[X_{\mathrm{Par}(e)},T\right]/\left(TP_e-1\right)$$

as $G_e=R_eT-1$. In particular h is not a zero divisor modulo (I_0,G_e).
Now it suffices to show that the ideals (I_0,G_e) and $(I_{\mathfrak{g}},G_e)$ of $A\left[\Gamma\right]\left[T\right]$ coincide. By construction $I_0\subset(I_{\mathfrak{g}},G_e)$. Conversely, the identities

$$\left\{\begin{array}{lllr} \Delta_{0,e_i} & = & R_{e_{i+1}}\delta_{0,i}-hX_{e_i}G_e & 0\leq i\leq n_e-2 \\ \Delta_{0,e'} & = & h\delta_{e'}+\left(X_{e_{i-1}}-lb\left(e_{i+1}\right)\right)^{-1}Q_{e'}\delta_{0,i} & \begin{array}{c} e'\in\mathbf{P}\left(\Gamma\right)\setminus\left(\downarrow e\right) \\ \mathrm{Anc}\left(e',e\right)=e_i\end{array} \end{array}\right. ,$$

guarantee that $\Delta_{0,e'}\in(I_0,G_e)$ for every $e'\in\mathbf{P}\left(\Gamma\right)$. In turn, this implies that $h\Delta_{e'',e'}\in(I_0,G_e)$ for every pair $(e',e'')\in\mathbf{P}\left(\Gamma\right)\times\mathrm{Anc}\left(e'\right)$ by virtue of (1.5). Since h is not a zero divisor modulo (I_0,G_e) we conclude that $\Delta_{e'',e'}\left(\mathfrak{g}\right)\in(I_0,G_e)$ for every such pair (e',e''). This completes the proof. □

Remark 2.7. Since $q_{\mathfrak{g}}$ is flat, (1.5) and (2.1) implies that $B_{\mathfrak{g}}$ is a sub-$\mathbb{C}$-algebra of $B_{\mathfrak{g}}\otimes_A A_h\simeq A_h\left[X_0\right]$. On the other hand, (1.5) and proposition 2.3 assert that it suffices to add the polynomials $\Delta_{e',e}\left(\mathfrak{g}\right)$, $(e,e')\in\mathbf{P}\left(\Gamma\right)\times\mathrm{Anc}_{\Gamma}\left(e\right)$, to the obvious ones $\Delta_{0,e}\left(\mathfrak{g}\right)$, $e\in\mathbf{P}\left(\Gamma\right)$, to guarantee that the fiber $q_{\mathfrak{g}}^{-1}\left(z_0\right)$ is the flat limit of the fibers $q_{\mathfrak{g}}^{-1}\left(z\right)$, $z\in Z_*$.

2. The fiber $q_{\mathfrak{g}}^{-1}\left(z_0\right)$.
Since $q_{\mathfrak{g}}:V_{\mathfrak{g}}\rightarrow Z$ is a flat morphism restricting to a trivial line bundle over Z_*, $V_{\mathfrak{g}}$ is a *GDS* over Z provided that the fiber $q_{\mathfrak{g}}^{-1}\left(z_0\right)$ is nonempty and reduced. To describe this fiber, we need the following auxiliary result.

Lemma 2.8. *If Child$\left(e_0\right)\neq\emptyset$ then there exists an isomorphism of $\mathbb{C}$-algebras*

$$A\left[\Gamma\right]/\left(h,I_{\mathfrak{g}}\right)\simeq\prod_{e\in Child\left(e_0\right)}\left(A\left[\Gamma\left(f\right)\right]/\left(h,I_{\mathfrak{g}(e)}\right)\right).$$

2.9. The polynomial $\Delta_{0,e_0}\in I_{\mathfrak{g}}$ reduces to Q_{e_0} modulo h. Since the roots $lb\left(e\right)\in\mathbb{C}$, $e\in\mathrm{Child}\left(e_0\right)$, of Q_{e_0} are simple, we deduce that

$$A\left[\Gamma\right]/\left(h,I_{\mathfrak{g}}\right)\ \simeq\ \prod_{e\in\mathrm{Child}\left(e_0\right)}\left(A\left[\Gamma\right]/\left(J_e,I_{\mathfrak{g}}\right)\right), \tag{2.2}$$

where $J_e=\left(h,X_0-lb\left(e\right)\right)$. Thus lemma 2.8 is a consequence of the following result.

Lemma 2.10. *For every $e\in$ Child$\left(e_0\right)$, the $\mathbb{C}$-algebras $A\left[\Gamma\right]/\left(J_e,I_{\mathfrak{g}}\right)$ and $A\left[\Gamma\left(e\right)\right]/\left(h,I_{\mathfrak{g}(e)}\right)$ are isomorphic.*

Proof. Since $X_{e_0} \notin N(\Gamma(e))$, we have

$$A[\Gamma(e)] = A[X_0] \otimes_A \mathbf{S}\left(M_{\Gamma(e)}\right) \simeq A[X_{e_0}] \otimes_A \mathbf{S}\left(M_{\Gamma(e)}\right).$$

With this choice of coordinates,

$$\begin{cases} Q_{e'}(\mathfrak{g}(e)) & = & Q_{e_0,e'}(\mathfrak{g}) & e' \in \mathbf{P}(\Gamma(e)) \\ Q_{e'',e'}(\mathfrak{g}(e)) & = & Q_{e'',e'}(\mathfrak{g}) & (e',e'') \in \mathbf{P}(\Gamma(e)) \times \mathrm{Anc}_{\Gamma(e)}(e') \end{cases}$$

We let $I_0 \subset A[\Gamma]$ be the ideal generated by the polynomials

$$\begin{cases} Q_{e_0,e'}(\mathfrak{g}) & e' \in \mathbf{P}(\Gamma(e)) \\ \Delta_{e'',e'}(\mathfrak{g}) & (e',e'') \in \mathbf{P}(\Gamma(e)) \times \left(\mathrm{Anc}_{\Gamma(e)}(e')\right) \\ \delta_{e_0,e'} = (lb(e) - lb(g))\, X_{e'} - X_{e_0} Q_{e_0,e'}(\mathfrak{g}) & \begin{cases} e' \in \mathbf{P}(\Gamma) \setminus (\{e_0\} \cup \mathbf{P}(\Gamma(e))) \\ g = \mathrm{Child}(e_0) \cap (\downarrow e') \neq e \end{cases} \end{cases}.$$

Since $\mathfrak{g}$ is fine-labelled, $lb(e) - lb(g) \in \mathbb{C}^*$ for every $g \in \mathrm{Child}(e_0) \setminus \{e\}$ and so, we can eliminate modulo I_0 all the variables $X_{e'}$ corresponding to nodes $e' \in \mathbf{P}(\Gamma) \setminus (\mathbf{P}(\Gamma(e)) \cup \{e_0\})$. We conclude that

$$A[\Gamma] / (J_e, I_0) = A[\Gamma(e)][X_0] / (h, X_0 - lb(e), I_0) \simeq A[\Gamma(e)] / \left(h, I_{\mathfrak{g}(e)}\right).$$

Now it suffices to show that the ideals (J_e, I_0) and $(J_e, I_{\mathfrak{g}})$ of $A[\Gamma]$ coincide.
If $e' \in \mathbf{P}(\Gamma) \setminus (\mathbf{P}(\Gamma(e)))$ then $\Delta_{0,e'}(\mathfrak{g}) \in J_e$ as $(X_0 - lb(e))$ divides $Q_{e'}(\mathfrak{g})$. If $e'' \in \mathrm{Anc}_{\Gamma}(e')$ then, letting $g' = \mathrm{Child}(e'') \cap (\downarrow e')$, we obtain that

$$\Delta_{e'',e'}(\mathfrak{g}) \mod J_e = \begin{cases} \delta_{e_0,e'} \in I_0 & \text{if } e'' = e_0 \\ \left(X_{\mathrm{Par}(e'')} - lb(g')\right) \delta_{e_0,e'} - \delta_{e_0,e''} Q_{e'',e'}(\mathfrak{g}) \in I_0 & \text{otherwise} \end{cases}.$$

If $e' \in \mathfrak{P}(\Gamma(e))$ then $\Delta_{0,e'}(\mathfrak{g})$ reduces to $S_{e_0}^{\{e\}}(lb(e))\, Q_{e_0,e'}(\mathfrak{g}) \in I_0$ modulo J_e. If $e'' \in \mathrm{Anc}_{\Gamma}(e')$ then either $e'' = e_0$ and $\Delta_{e_0,e'}(\mathfrak{g})$ reduces to $Q_{e_0,e'}(\mathfrak{g}) \in I_0$ modulo J_e, or $e'' \neq e_0$ and $\Delta_{e'',e'}(\mathfrak{g}) \in I_0$. This proves that $(J_e, I_{\mathfrak{g}}) = (J_e, I_0)$. □

The following proposition completes the proof of theorem 2.1.

Proposition 2.11. *The fiber $q_{\mathfrak{g}}^{-1}(z_0)$ of $q_{\mathfrak{g}} : V_{\mathfrak{g}} \to Z$ is nonempty and reduced, consisting of the disjoint union of curves $C_e \simeq Spec\left(\mathbb{C}\left[X_{Par(e)}\right]\right)$, $e \in Leaves(\Gamma)$, with defining ideals*

$$I_{\mathfrak{g}}(e) = \left(I_{\mathfrak{g}}, h, \left(X_{Par^{k+1}(e)} - lb\left(Par^{k-1}(e)\right)\right)_{1 \leq k \leq n_e}\right) \subset A[\Gamma].$$

Proof. We proceed by induction on the height $h(\Gamma)$ of Γ. If $h(\Gamma) = 0$ then $V_{\mathfrak{g}} = \mathrm{Spec}(A[X_0])$ and $q_{\mathfrak{g}}^{-1}(z_0) \simeq \mathrm{Spec}(\mathbb{C}[X_0])$ is reduced. Otherwise, if $\mathrm{Child}(e_0) \neq \emptyset$ then lemma 2.8 implies that

$$q_{\mathfrak{g}}^{-1}(z_0) \simeq \mathrm{Spec}(A[\Gamma] / (h, I_{\mathfrak{g}})) \simeq \bigsqcup_{e' \in \mathrm{Child}(e_0)} \mathrm{Spec}\left(A\left[\Gamma(e')\right] / \left(h, I_{\mathfrak{g}(e')}\right)\right)$$

decomposes as the disjoint union of curves $D_{e'}$ isomorphic to the fibers $q_{\mathfrak{g}(e')}^{-1}(z_0)$ of $q_{\mathfrak{g}(e')} : V_{\mathfrak{g}(e')} \to Z$, $e' \in \mathrm{Child}(e_0)$. Since $\Gamma(e')$ has height $h(\Gamma) - 1$, these fibers are nonempty and reduced, whence $q_{\mathfrak{g}}^{-1}(z_0)$ is. In view of the isomorphism (2.2), the description of the irreducible components of $q_{\mathfrak{g}}^{-1}(z_0)$ follows easily by induction. □

Corollary 2.12. *For every fine-labelled rooted tree $\mathfrak{g}$, the surface $V_{\mathfrak{g}}$ is nonsingular.*

Proof. Indeed $q_{\mathfrak{g}} : V_{\mathfrak{g}} \to Z$ is a flat morphism with regular geometric fibers, hence a smooth morphism. Thus $V_{\mathfrak{g}}$ is regular as $Z \simeq \mathbb{A}^1_{\mathbb{C}}$ is. □

3. Embeddings of GDS's in affine spaces

In this section, we consider GDS's $q : V \to Z = \mathrm{Spec}\,(A)$ such that $q^{-1}(z)$ is integral for every $z \in Z_*$. We prove the following result.

Theorem 3.1. *Every GDS $q : V \to Z$ is Z-isomorphic to a GDS $q_{\mathfrak{g}} : V_{\mathfrak{g}} \to Z$ for a certain fine-labelled rooted tree $\mathfrak{g} = (\Gamma, lb)$.*

The proof divides as follows. In the first subsection we recall [5] how to attach a weighted rooted tree $\gamma = (\Gamma, w)$ to a pair $\left(q : V \to Z, \phi : V \to \mathbb{A}^1_Z\right)$ consisting of a GDS $q : V \to Z$ and a Z-morphism $\phi : V \to \mathbb{A}^1_Z$ restricting to an isomorphism over Z_*. In turn, this weighted rooted tree γ defines a GDS $q_\gamma : W_\gamma \to Z$ which is Z-isomorphic to V. In the second subsection, given a weighted rooted tree $\gamma = (\Gamma, w)$ we construct a fine-labelled tree $\mathfrak{g}_w = (\Gamma, lb_w)$ with the same underlying tree Γ and a closed embedding $i_\gamma : W_\gamma \hookrightarrow \mathrm{Spec}\,(A[\Gamma])$ inducing a Z-isomorphism $\psi_\gamma : W_\gamma \xrightarrow{\sim} V_{\mathfrak{g}_w}$.

From GDS's to weighted rooted trees.
In this subsection, we freely use the description of GDS's given in [5]. We recall without proof how to associate a weighted rooted tree $\gamma = (\Gamma, w)$ to a GDS $q : V \to Z$. We also recall how such a weighted rooted tree γ defines a GDS $q_\gamma : W_\gamma \to Z$ which comes equipped with a canonical Z-morphism $\phi_{\gamma,0} : W_\gamma \to \mathbb{A}^1_Z$ restricting to an isomorphism over Z_*.

3.2. Given a GDS $q : V = \mathrm{Spec}\,(B) \to Z$, q restricts to the trivial line bundle over $Z_* = \mathrm{Spec}\,(A_h)$. Since q is flat, we can find an isomorphism $A_h[T] \simeq B \otimes_A A_h$ which extends to an injection $A[T] \hookrightarrow B$. The corresponding Z-morphism $\phi_0 : V \to \mathbb{A}^1_Z = \mathrm{Spec}\,(A[T])$ restricts to an isomorphism over Z_*. This data determines a weighted rooted tree $\gamma = (\Gamma, w)$ as follows.

3.3. If $\phi_0 : V \to \mathbb{A}^1_Z$ is an isomorphism then we let γ be the trivial tree with one element e_0. Otherwise, ϕ_0 is constant on the irreducible components $C_1, \ldots, C_r$ of the fiber $q^{-1}(z_0)$, and the open subsets

$$V_i = \left(V \setminus q^{-1}(z_0)\right) \cup C_i \quad 1 \leq i \leq r$$

are Z-isomorphic to $\mathbb{A}^1_Z$. Therefore, there exist $n_i \geq 1$, a polynomial $\sigma_i \in A = \mathbb{C}[h]$ of degree $\deg_h(\sigma_i) < n_i$ and an isomorphism $\tau_i : H^0(Z, q_*\mathcal{O}_{V_i}) \xrightarrow{\sim} A[T_i]$ such that $\phi_0 \mid_{V_i}$ is induced by the polynomial $h^{n_i}T_i + \sigma_i \in A[T_i]$, $1 \leq i \leq r$. Note that over Z_*, the transition isomorphism $\tau_j \circ \tau_i^{-1}$ is given by the A_h-algebras isomorphism

$$A_h[T_i] \xrightarrow{\sim} A_h[T_j], T_i \mapsto h^{n_j - n_i}T_j + h^{-n_i}(\sigma_j - \sigma_i)$$

(compare with [3] and [6]). Letting

$$\sigma_i = \sum_{k=0}^{n_i - 1} w_{i,k} h^k \in \mathbb{C}[h] = A, \quad 1 \leq i \leq r,$$

we consider the chains $(C_i, w_i) = \{e_{i,0} < e_{i,1} < \cdots < e_{i,n_i}\}$ with the weights

$$w_i([e_{i,k}, e_{i,k+1}]) = w_{i,k} \quad 1 \leq i \leq r, 0 \leq k \leq n_i - 1.$$

Since $q : V \to Z$ is affine, proposition 2.6 in [5] implies that $n_{ij} = \mathrm{ord}_{z_0}(\sigma_j - \sigma_i) < \min(n_i, n_j)$. Consequently, there exist isomorphisms of weighted subchain

$$\theta_{ij} : C_{ij} = \left(\downarrow e_{i,n_{ij}}\right)_{C_i} \xrightarrow{\sim} C_{ji} = \left(\downarrow e_{j,n_{ij}}\right)_{C_j} \quad i \neq j, 1 \leq i, j \leq r,$$

and a unique weighted rooted tree $\gamma = (\Gamma, w)$ with the leaves f_i, together with isomorphisms of weighted chains $\theta_i : (C_i, w_i) \simeq (\downarrow f_i)_\Gamma$, $1 \le i \le r$, such that $\theta_i = \theta_j \circ \theta_{ij}$ on C_{ij} (see [5, Proposition 1.12]).

3.4. Starting with a weighted rooted tree $\gamma = (\Gamma, w)$, we construct a *GDS* $q_\gamma : W_\gamma \to Z$ as follows. If Γ is the trivial tree with one element e_0 then we let $W_\gamma = \mathrm{Spec}\,(A\,[X_0]) \simeq \mathbb{A}^1_Z$ and $q_\gamma = pr_Z$. Otherwise, we let $r\,(\gamma) = \mathrm{Card}\,(\mathrm{Leaves}\,(\Gamma)) \ge 1$, and we consider the maximal chains

$$(\downarrow f)_\Gamma = \left\{ e_{f,0} = e_0 < e_{f,1} < \cdots < e_{f,n_f-1} < e_{f,n_f} = f \right\}, \quad f \in \mathrm{Leaves}\,(\Gamma)\,. \tag{3.1}$$

The prescheme $\pi_\gamma : X_\gamma \to Z$ obtained from Z by replacing the closed point z_0 by $r\,(\gamma)$ points x_f, $f \in \mathrm{Leaves}\,(\Gamma)$, admits a covering $\mathcal{U}_\gamma$ by the open subsets $X_f = \pi_\gamma^{-1}(Z_*) \cup \{x_f\} \simeq Z$, $f \in \mathrm{Leaves}\,(\Gamma)$. Moreover $X_{ff'} = X_f \cap X_{f'} \simeq Z_*$ for every two different leaves f and f'. We let $\mathcal{L}_\gamma$ be the sub-$\mathcal{O}_X$-module of $\mathcal{K}\,(X_\gamma)$ generated by $(h \circ \pi_\gamma)^{-n_f}$ on X_f, $f \in \mathrm{Leaves}\,(\Gamma)$, and we denote by $s_\gamma \in \Gamma\,(X, \mathcal{L}_\gamma)$ the canonical section of $\mathcal{L}_\gamma$ corresponding to the constant section 1 of $\mathcal{K}\,(X_\gamma)$. We let $\sigma\,(\gamma) = \{\sigma_f\,(\gamma)\}_{f \in \mathrm{Leaves}(\Gamma)} \in C^0\left(\mathcal{U}_\gamma, \mathcal{O}_{X_\gamma}\right)$ be the 0-cochain defined by

$$\sigma_f\,(\gamma) \;=\; \sum_{j=0}^{n_f-1} w\,([e_{f,j}, e_{f,j+1}])h^j \in A \simeq H^0\left(X_f, \mathcal{O}_{X_\gamma}\right), \quad f \in \mathrm{Leaves}\,(\Gamma)\,.$$

Since s_γ does not vanish on $X_{ff'} \simeq Z_*$, $f \neq f'$, $f, f' \in \mathrm{Leaves}\,(\Gamma)$, there exists a unique Čech 1-cocycle $\left\{g_{ff'}\right\}_{f,f' \in \mathrm{Leaves}(\Gamma)} \in C^1\left(\mathcal{U}_\gamma, \mathcal{L}_\gamma^\vee\right)$ such that

$$g_{ff'} \circ s_\gamma = \sigma_{f'}\,(\gamma)\,|_{X_{ff'}} - \sigma_f\,(\gamma)\,|_{X_{ff'}}, \quad f \neq f', f, f' \in \mathrm{Leaves}\,(\Gamma)\,.$$

3.5. Now there exists a unique quasicoherent $\mathcal{O}_{X_\gamma}$-algebra $\mathcal{A}_\gamma$, together with isomorphisms $\tau_f : \mathcal{A}_\gamma\,|_{X_f} \xrightarrow{\sim} \mathbf{S}\left(\mathcal{L}_\gamma\,|_{X_f}\right)$, $f \in \mathrm{Leaves}\,(\Gamma)$, such that over the overlaps $X_{ff'}$, the $\mathcal{O}_{X_{ff'}}$-algebras isomorphisms $\tau_{f'} \circ \tau_f^{-1} : \mathbf{S}\left(\mathcal{L}_\gamma\,|_{X_{ff'}}\right) \xrightarrow{\sim} \mathbf{S}\left(\mathcal{L}_\gamma\,|_{X_{ff'}}\right)$ are induced by the $\mathcal{O}_{X_{ff'}}$-modules homomorphisms

$$\left(g_{ff'}, \mathrm{Id}\right) : \mathcal{L}_\gamma\,|_{X_{ff'}} \to \mathcal{O}_{X_{ff'}} \oplus \mathcal{L}_\gamma\,|_{X_{ff'}} \subset \mathbf{S}\left(\mathcal{L}_\gamma\,|_{X_{ff'}}\right). \tag{3.2}$$

By theorem 3.3 in [5], the morphism

$$q_\gamma = \pi_\gamma \circ \rho_\gamma : W_\gamma = \mathbf{Spec}\,(\mathcal{A}_\gamma) \xrightarrow{\rho_\gamma} X_\gamma \xrightarrow{\pi_\gamma} Z$$

is a *GDS* over Z. It restricts to the trivial line bundle over Z_*, whereas $q_\gamma^{-1}(z_0)$ is the disjoint union of the curves $C_f = \rho_\gamma^{-1}(x_f) \simeq \mathbb{A}^1_{\mathbb{C}}$, $f \in \mathrm{Leaves}\,(\Gamma)$.

3.6. This *GDS* $q_\gamma : W_\gamma \to Z$ comes equipped with the canonical section $\phi_{\gamma,0} \in H^0\left(Z, (q_\gamma)_*\,\mathcal{O}_{W_\gamma}\right)$ whose restriction to $W_\gamma\,|_{X_f}$, $f \in \mathrm{Leaves}\,(\Gamma)$, is simply given by

$$\phi_{\gamma,0}\,|_{X_f} = \left(\sigma_f\,(\gamma)\,, s_\gamma\,|_{X_f}\right) \in H^0\left(X_f, \mathcal{A}_\gamma\right) \simeq H^0\left(X_f, \mathbf{S}\,(\mathcal{L}_\gamma)\right).$$

Since $s_\gamma$does not vanish on $\pi_\gamma^{-1}(Z_*) \subset X_\gamma$, the corresponding Z-morphism $\phi_{\gamma,0} : W_\gamma \to \mathbb{A}^1_Z$ restricts to an isomorphism over Z_*. If γ is obtained from a pair $\left(q : V \to Z, \phi : V \to \mathbb{A}^1_Z\right)$ by the procedure described in 3.3 then, by construction, there exists a Z-isomorphism $\psi : V \xrightarrow{\sim} W_\gamma$ such that $\phi_0 = \phi_{\gamma,0} \circ \psi$. This leads to the following result.

Theorem 3.7. [5, Theorem 5.1] *For every GDS $q : V \to Z$ there exists a weighted rooted tree $\gamma = (\Gamma, w)$ and a Z-isomorphism $\psi : V \xrightarrow{\sim} W_\gamma$.*

Example 3.8. The nonsingular surface $V \subset \operatorname{Spec}(\mathbb{C}[x,y,z])$ with equation

$$x^2 z = y^2 - 2xy - 1$$

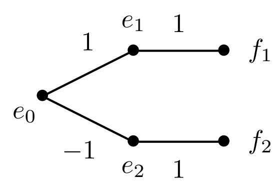
is a *GDS* over $Z = \operatorname{Spec}(\mathbb{C}[x])$ for the morphism $pr_x : V \to \operatorname{Spec}(\mathbb{C}[x])$. Letting $z_0 = (x) \in Z$, we get $V \times_Z Z_* \simeq \operatorname{Spec}\left(\mathbb{C}\left[x, x^{-1}\right][y]\right)$, whereas $pr_x^{-1}(z_0)$ is the disjoint union of curves

$$C_1 = \{x = 0, y = 1\} \cap V \quad \text{and} \quad C_2 = \{x = 0, y = -1\} \cap V.$$

isomorphic to $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec}(\mathbb{C}[z])$. The projection $pr_{x,y} : V \to \mathbb{A}^1_Z = \operatorname{Spec}(\mathbb{C}[x][y])$ is a birational Z-morphism restricting to an isomorphism over Z_*. The rational functions $T_1 = x^{-2}(y-1) - x^{-1}$ and $T_2 = x^{-2}(y+1) - x^{-1}$ on V induce Z-isomorphisms

$$\left(V \setminus pr_x^{-1}(z_0)\right) \cup C_1 \simeq \operatorname{Spec}(\mathbb{C}[x][T_1]) \quad \text{and} \quad \left(V \setminus pr_x^{-1}(z_0)\right) \cup C_2 \simeq \operatorname{Spec}(\mathbb{C}[x][T_2])$$

Therefore, the pair $\left(pr_x : V \to Z, pr_{x,y} : V \to \mathbb{A}^1_Z\right)$ corresponds to the following weighted rooted tree

$\gamma =$ 

Embedding of a *GDS* W_γ in an affine space.
Given a *GDS* $q_\gamma : W_\gamma \to Z$ defined by a weighted rooted tree $\gamma = (\Gamma, w)$, we construct in 3.9-3.17 below a fine-labelled tree $\mathfrak{g} = (\Gamma, lb_w)$ with the same underlying rooted tree Γ, and a collection of regular functions

$$\left(\phi_{\gamma,0}, (\phi_{\gamma,e})_{e \in \mathbf{P}(\Gamma)}\right) \in B_\gamma = H^0\left(Z, (q_\gamma)_* \mathcal{O}_{W_\gamma}\right)$$

defining a closed embedding $i_\gamma : W_\gamma \hookrightarrow \mathbb{A}^{d(\Gamma)+1}_Z = \operatorname{Spec}(A[\Gamma])$ with image $V_{\mathfrak{g}_w}$.

3.9. Consider the canonical section $\phi_{\gamma,0} \in B_\gamma$ defined in 3.6. If Γ is the trivial tree with one element e_0 then

$$W_\gamma \simeq \operatorname{Spec}(A[\phi_{\gamma,0}]) \simeq \operatorname{Spec}(A[\Gamma]).$$

Otherwise, for every $f \in \operatorname{Leaves}(\Gamma)$, the multiplication by h^{n_f} gives rise to an $\mathcal{O}_{X_f}$-modules isomorphism $\mathcal{L}_\gamma \mid_{X_f} \xrightarrow{\sim} \mathcal{O}_{X_f}$, whence to an A-algebras isomorphism $\tau_f : H^0(X_f, \mathcal{A}_\gamma) \xrightarrow{\sim} A[T]$ such that

$$\tau_f(\phi_{\gamma,0}) \quad = \quad h^{n_f} T + \sigma_f(\gamma) = \sum_{k=0}^{n_f} w_{f,k} h^k \in A[T], \tag{3.3}$$

where $w_{f,n_f} = T$. Since $n_f \geq 1$, we deduce that $\phi_{\gamma,0}$ is locally constant on $q_\gamma^{-1}(z_0)$, with the value $w_{f,0} \in \mathbb{C}$ on $C_f = \rho_\gamma^{-1}(x_f) \simeq \operatorname{Spec}(\mathbb{C}[T])$. For every $e \in N(\Gamma)$, we let

$$F(e) = \bigcup_{f \in \operatorname{Leaves}(\Gamma(e))} C_f \subset q_\gamma^{-1}(z_0).$$

In particular, $F(e_0) = q_\gamma^{-1}(z_0)$ and $F(f) = C_f$ for every $f \in \mathrm{Leaves}(\Gamma)$. We have the following result.

Proposition 3.10. *If $h(\Gamma) \geq 1$ then there exists a fine-labelled rooted tree $\mathfrak{g}_w = (\Gamma, lb_w)$ with underlying tree Γ, and a collection of regular functions $\phi_{\gamma,Par(e_0)} = \phi_{\gamma,0}, \phi_{\gamma,e} \in B_\gamma$, $e \in \mathbf{P}(\Gamma)$, such that the following hold.*

1) For every $e \in \mathbf{P}(\Gamma)$ with $(\downarrow e)_\Gamma = \{e_0 < e_1 < \cdots < e_n = e\}$,

$$\phi_{\gamma,e} = h^{-1} Q_e(\mathfrak{g}_w)\left(\phi_{\gamma,0}, \phi_{\gamma,e_0}, \ldots, \phi_{\gamma,e_{n-1}}\right).$$

2) Letting Child$(e) = \{g_1, \ldots, g_r\}$, $\phi_{\gamma,Par(e)}$ is constant on $F(g_i)$ with the value $lb_w(g_i) \in \mathbb{C}$ whereas $\phi_{\gamma,e}$ restricts to a coordinate function on $C_g \subset q_\gamma^{-1}(z_0)$ for every $g \in$ Child$(e) \cap$ Leaves(Γ).

3.11. Starting with $\phi_{\gamma,0}$, we construct by induction the labelling $lb_w : N(\Gamma) \setminus \{e_0\} \to \mathbb{C}$ and the regular functions $\phi_{\gamma,e}$, $e \in \mathbf{P}(\Gamma)$. More precisely, for every $n \leq h(\Gamma)$ in $N(\Gamma)$, we define the labelling function lb_w on $N_n(\Gamma)$ in terms of the functions $\phi_{\gamma,e'}$ corresponding to nodes $e' \in N_k(\Gamma)$, $k \leq n-2$. Then we define the functions $\phi_{\gamma,e}$ for nodes $e \in N_{n-1}(\Gamma)$ in terms of those functions $\phi_{\gamma,e'}$ and the values of lb_w on the nodes $e'' \in N_k(\Gamma)$, $k \leq n$.

3.12. Step 1. Given $g \in \mathrm{Child}(e_0)$, we let

$$lb_w(g) := \phi_{\gamma,0}\mid_{F(g)} = w_{f,0},$$

where f is any leaf of $\Gamma(g)$. This is well-defined as $g = e_{f,1}$ for every such leaf f (see (3.1)). If g' is another child of e_0 then there exists $f' \in \mathrm{Leaves}(\Gamma(g'))$ such that $g' = e_{f',1}$, and hence

$$lb_w(g) = w_{f,0} \neq w_{f',0} = lb_w\left(g'\right)$$

by definition 1.4. This defines the labelling function lb_w on the nodes at level 1. Clearly,

$$\phi_{\gamma,e_0} : \;=\; h^{-1} Q_{e_0}(\mathfrak{g}_w)(\phi_{\gamma,0}) = h^{-1} \prod_{g \in \mathrm{Child}(e_0)} (\phi_{\gamma,0} - lb_w(g)) \in B_\gamma \otimes_A A_h$$

is a regular function on W_γ. Letting

$$s_g = S_{e_0}^{\{g\}}(\mathfrak{g}_w)(lb_w(g)) = \prod_{g' \in \mathrm{Child}(e_0) \setminus \{g\}} \left(lb_w(g) - lb_w\left(g'\right)\right) \in \mathbb{C}^*, \quad g \in \mathrm{Child}(e_0),$$

we deduce from Taylor's Formula that for every $f \in \mathrm{Leaves}(\Gamma(g))$ there exists $P_0(g,f) \in A[T]$ such that

$$\tau_f(\phi_{\gamma,e_0}) = s_g w_{f,1} + h P_0(g,f) \in A[T].$$

If g is a leaf of Γ then $w_{g,1} = T$ (see (3.3)) and so, ϕ_{γ,e_0} restricts to a coordinate function on $C_g \subset q_\gamma^{-1}(z_0)$. Thus (1)-(2) hold for $e_0 \in N(\Gamma)$.

3.13. Step 2. To define lb_w on the nodes at level 2, we proceed as follows. Given $g' \in \mathrm{Child}(g)$, ϕ_{γ,e_0} is constant on $F(g') \subset q_\gamma^{-1}(z_0)$ with the value

$$lb_w\left(g'\right) := s_g w_{f,1} \in \mathbb{C},$$

where f is any leaf of $\Gamma(g')$. Given another child g'' of g and a leaf f' of $\Gamma(g'')$, $g = e_{f,1} = e_{f',1}$ is the first common ancestor of f and f'. Thus $e_{f,2} \neq e_{f',2}$, whence $w_{f,1} \neq w_{f',1}$ as γ is a weighted rooted tree. Since $s_g \in \mathbb{C}^*$ we deduce that $lb_w(g') \neq lb_w(g'')$. This defines the labelling function lb_w on the nodes at level 2.

Suppose that the labelling lb_w has been defined for every node e at level $k \leq n+1$ and that for every $e \in \mathbf{P}(\Gamma) \cap N_n(\Gamma)$ with $(\downarrow e) = \{e_0 < e_1 < \cdots < e_n = e\}$, the regular functions $\phi_{e_i} \in B_\gamma$, $0 \leq i \leq n-1$ have been constructed, satisfying (1) and (2) of proposition 3.10. Since lb_w is defined on $N_k(\Gamma)$, $k \leq n+1$, the polynomial $Q_e(\mathfrak{g}_w) \in \mathbb{C}\left[X_0, X_{e_0}, \ldots, X_{e_{n-1}}\right]$ is well-defined (see 1.9). We have the following result.

Lemma 3.14. *If e is not a leaf of Γ then*

$$\phi_{\gamma,e_n} = h^{-1} Q_e(\mathfrak{g}_w)\left(\phi_{\gamma,0}, \phi_{\gamma,e_0}, \ldots, \phi_{\gamma,e_{n-1}}\right) \in B_\gamma \otimes_A A_h$$

is a regular function on W_γ satisfying (1)-(2) in proposition 3.10.

Proof. In view of the definition of the functions ϕ_{γ,e_i}, $0 \leq i \leq n-1$, we deduce from Taylor's Formula that for every $g \in \mathrm{Child}(e)$ and every $f \in \mathrm{Leaves}(\Gamma(g))$, there exists $P_{n-1}(g,f) \in A[T]$ such that

$$\tau_f\left(\phi_{\gamma,e_{n-1}}\right) = lb_w(g) + \left(\prod_{i=0}^{n-1} s_{e_i}^{n-1-i} w_{f,n+1} + Q_{n-1,g}(w_{f,0}, \ldots, w_{f,n})\right) h + h^2 P_{n-1}(g,f) \in A[T],$$

where $Q_{n-1,g} \in \mathbb{C}[Y_0, \ldots, Y_n]$ is independent on the choice of a leaf f of $\Gamma(g)$ and

$$s_{e_i} = S_{e_i}^{\{e_{i+1}\}}(\mathfrak{g}_w)(lb_w(e_{i+1})) \in \mathbb{C}^*, \quad 1 \leq i \leq n-1.$$

By definition,

$$\phi_{\gamma,e_n} = h^{-1} S_{e_n}(\mathfrak{g}_w)\left(\phi_{\gamma,e_{n-1}}\right) \prod_{i=0}^{n-1} S_{e_i}^{\{e_{i+1}\}}(\mathfrak{g}_w)\left(\phi_{\gamma,e_{i-1}}\right) \in B_\gamma \otimes_A A_h$$

and $\phi_{\gamma,e_{i-1}} |_{C(f)} = lb_w(e_{i+1})$, $0 \leq i \leq n$. Again, Taylor's Formula implies that there exists $P_n(g,f) \in A[T]$ such that

$$\tau_f(\phi_{\gamma,e_n}) = (\lambda_g w_{f,n+1} + \mu_g) + hP_n(g,f) \in A[T], \tag{3.4}$$

where

$$\begin{cases} \lambda_g = S_{e_n}^{\{g\}}(\mathfrak{g}_w)(lb_w(g)) \left(\prod_{i=0}^{n-1} s_{e_i}^{n-i}\right) \in \mathbb{C}^*, \\ \mu_g = S_{e_n}^{\{g\}}(\mathfrak{g}_w)(lb_w(g)) \left(\prod_{i=0}^{n-1} s_{e_i}\right) Q_{n-1,g}(w_{f,0}, \ldots, w_{f,n}) \in \mathbb{C}. \end{cases}$$

Thus ϕ_{γ,e_n} is regular on $W_\gamma |_{X_f}$ for every $f \in \mathrm{Leaves}(\Gamma(e_n))$. Given $f' \in \mathrm{Leaves}(\Gamma) \setminus \mathrm{Leaves}(\Gamma(e_n))$, there exists an indice $k \leq n-1$ such that e_k is the first common ancestor of f' and e_n. Letting $e' = \mathrm{Child}(e_k) \cap (\downarrow f')$, $\left(X_{e_{k-1}} - lb_w(e')\right)$ divides $Q_{e_n}(\mathfrak{g}_w)$ and so, $\tau_{f'}\left(\phi_{\gamma,e_{k-1}} - lb_w(e')\right) \in hA[T]$. In turn, this implies that $\tau_{f'}(\phi_{\gamma,e_n}) \in A[T]$. Thus ϕ_{γ,e_n} is a regular function on W_γ. If $g \in \mathrm{Child}(e_n)$ is a leaf of Γ then $w_{g,n+1} = T$ (see (3.3)) and so, ϕ_{γ,e_n} restricts to a coordinate function on $C_g \subset q_\gamma^{-1}(z_0)$. □

3.15. Consider the regular function ϕ_{γ,e_n} constructed above. Given $g \in \mathrm{Child}(e_n) \setminus \mathrm{Leaves}(\Gamma)$ and $g' \in \mathrm{Child}(g)$, we deduce from (3.4) that ϕ_{γ,e_n} is constant on $F(g') \subset q_\gamma^{-1}(z_0)$ with the value

$$lb_w(g') := \lambda_g w_{f,n+1} + \mu_g, \tag{3.5}$$

where f is any leaf of $\Gamma(g')$. Since neither λ_g nor μ_g depend on the choice of $f \in \mathrm{Leaves}(\Gamma(g))$, the same argument as in 3.12 shows that $lb_w(g') \neq lb_w(g'')$ for every two distinct children g'

and g'' of g. In this way, we define the labelling function lb_w on the nodes of Γ at level $n+2$. Now the proof of proposition 3.10 can be completed by induction.

Remark 3.16. By construction, the labelling $lb_w : N(\Gamma) \to \mathbb{C}$ is uniquely determined by the weight function $w : E(\Gamma) \to \mathbb{C}$. Conversely, since $lb_w(e) = w([e_0, e]_\Gamma)$ for every node $e \in N_1(\Gamma)$ (see 3.12), we deduce from (3.5) that the weight function $w : E(\Gamma) \to \mathbb{C}$ can be recursively recovered from the associated labelling lb_w.

The following result completes the proof of theorem 3.1.

Theorem 3.17. *For every GDS $q_\gamma : W_\gamma \to Z = Spec(A)$ defined from a weighted rooted tree $\gamma = (\Gamma, w)$, the A-algebras homomorphism*

$$A[\Gamma] \to B_\gamma, \begin{cases} X_0 \mapsto \phi_{\gamma,0} \\ X_e \mapsto \phi_{\gamma,e} \quad e \in \mathbf{P}(\Gamma) \end{cases}$$

is surjective, with kernel $I_{\mathfrak{g}_w}$. In other words, the functions $\phi_{\gamma,0}, (\phi_{\gamma,e})_{e\in\mathbf{P}(\Gamma)}$ define a closed embedding $i_\gamma : W_\gamma \hookrightarrow \mathbb{A}_Z^{d(\Gamma)+1} = Spec(A[\Gamma])$ inducing a Z-isomorphism $\psi_\gamma : W_\gamma \xrightarrow{\sim} V_{\mathfrak{g}_w}$.

Proof. By construction, $W_\gamma \times_Z Z_* \simeq \mathrm{Spec}(A_h[\phi_{\gamma,0}])$. Given two different leaves f and f' of Γ with first common ancestor $e \in \mathbf{P}(\Gamma)$, it follows from proposition 3.10 that $\phi_{\gamma,\mathrm{Par}(e)}$ takes disctint values on the curves C_f and $C_{f'}$. Thus i_γ separates the irreducible components of $q_\gamma^{-1}(z_0)$. If $f \in \mathrm{Leaves}(\Gamma)$ then $\phi_{\gamma,\mathrm{Par}(f)}$ restricts to a coordinate function on C_f. This proves that i_γ is a closed embedding of Z-schemes. By (1) in proposition 3.10, the relation

$$\Delta_{0,e}(\mathfrak{g}_w)\left(\phi_{\gamma,0}, (\phi_{\gamma,e})_{e\in\mathfrak{P}(\Gamma)}\right) = h\phi_{\gamma,e} - Q_e(\mathfrak{g}_w)\left(\phi_{\gamma,0}, (\phi_{\gamma,e})_{e\in\mathfrak{P}(\Gamma)}\right) = 0$$

holds in B_γ for every $e \in \mathbf{P}(\Gamma)$. In turn, (1.5) implies that

$$h\Delta_{e',e}(\mathfrak{g}_w)\left(\phi_{\gamma,0}, (\phi_{\gamma,e})_{e\in\mathfrak{P}(\Gamma)}\right) = 0 \quad (e, e') \in \mathbf{P}(\Gamma) \times \mathrm{Anc}_\Gamma(e)$$

and so, $\Delta_{e',e}(\mathfrak{g}_w)\left(\phi_0, (\phi_e)_{e\in\mathfrak{P}(\Gamma)}\right) = 0$ as B_γ is an integral A-algebra. Since i_γ maps the irreducible components of $q_\gamma^{-1}(z_0)$ bijectively on the irreducible components of $q_{\mathfrak{g}_w}^{-1}(z_0)$ (see proposition 2.11), we deduce that i_γ induces a bijective morphism $\psi_\gamma : W_\gamma \to V_{\mathfrak{g}_w}$. By virtue of Zariski's Main Theorem, ψ_γ is an isomorphism as $V_{\mathfrak{g}_w}$ is nonsingular (see corollary 2.12), whence normal. □

Example 3.18. We again consider the surface $V \subset \mathrm{Spec}(\mathbb{C}[x,y,z])$ with equation

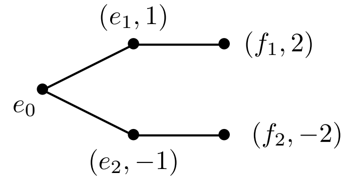
$$x^2 z = y^2 - 2xy - 1$$

introduced in example 3.8. Starting with the weighted rooted tree $\gamma = (\Gamma, w)$ corresponding to the pair

$$\left(pr_x : V \to Z, pr_{x,y} : V \to \mathbb{A}_Z^1\right),$$

the above procedure gives the following fine-labelled rooted tree

$\mathfrak{g}_w = (\Gamma, lb_w) =$



Thus $V_{\mathfrak{g}_w} \subset \mathrm{Spec}\left(A\left[X_0\right]\left[X_{e_0}, X_{e_1}, X_{e_2}\right]\right)$ is the surface with equations

$$\left\{\begin{array}{c} hX_{e_0} = \left(X_0^2 - 1\right), \quad hX_{e_1} = \left(X_0 + 1\right)\left(X_{e_0} - 2\right), \quad hX_{e_2} = \left(X_0 - 1\right)\left(X_{e_0} + 2\right) \\ \left(X_0 - 1\right) X_{e_1} = X_{e_0}\left(X_{e_0} - 2\right), \quad \left(X_0 + 1\right) X_{e_2} = X_{e_0}\left(X_{e_0} + 2\right) \end{array}\right. .$$

This shows that in general, the embedding of a GDS V obtained by the above procedure is not "the best possible". However, note that the $\mathbb{C}$-algebras homomorphism

$$A\left[X_0, X_{e_0}, X_{e_1}, X_{e_2}\right] \to \mathbb{C}\left[x, x^{-1}y, z\right], \left\{\begin{array}{l} h \mapsto x, \quad X_0 \mapsto y \\ X_{e_0} \mapsto xz + 2y \\ X_{e_1} \mapsto x^{-1}\left(xz + 2y - 2\right)\left(y + 1\right) \\ X_{e_2} \mapsto x^{-1}\left(xz + 2y + 2\right)\left(y - 1\right) \end{array}\right.$$

induces an isomorphism $V \xrightarrow{\sim} V_{\mathfrak{g}}$.

The following result is complementary to proposition 3.10.

Proposition 3.19. *For every GDS $q_{\mathfrak{g}} : V_{\mathfrak{g}} \to Z$ defined by a fine-labelled rooted tree $\mathfrak{g} = (\Gamma, lb)$, the weighted tree γ associated to the pair $\left(q_{\mathfrak{g}} : V_{\mathfrak{g}} \to Z, pr_{X_0} : V_{\mathfrak{g}} \to \mathbb{A}_Z^1\right)$ has the same underlying tree Γ as $\mathfrak{g}$, and the GDS $V_{\mathfrak{g}}$ is Z-isomorphic to W_γ via the closed embedding $i_\gamma : W_\gamma \hookrightarrow \mathbb{A}_Z^{d(\Gamma)+1}$.*

Proof. By construction of the embedding $i_\gamma : W_\gamma \hookrightarrow \mathbb{A}_Z^{d(\Gamma)+1}$, the canonical Z-morphism $\phi_{0,\gamma} : W_\gamma \to \mathbb{A}_Z^1$ factors as $\phi_{\gamma,0} = pr_{X_0} \circ i_\gamma$ (see 3.17). So the statement follows from remark 3.16. □

4. $\mathbb{G}_{a,Z}$-actions on GDS's over Z

In this subsection we study $\mathbb{G}_{a,Z}$-actions on a GDS $q : V \to Z$. In view of the well known correspondence between algebraic $\mathbb{G}_{a,Z}$-actions on V and locally nilpotent A-derivations of the algebra $B = H^0\left(Z, q_*\mathcal{O}_V\right)$, this is the same as to study the set $\mathrm{LND}_A(B)$ of these locally nilpotent derivations.

$\mathbb{G}_{a,Z}$-actions on the GDS's $q_\gamma : W_\gamma \to Z$.
Here we recall following [5] the description of $\mathbb{G}_{a,Z}$-actions on a GDS $q_\gamma : W_\gamma \to Z$ defined from a weighted rooted tree $\gamma = (\Gamma, w)$. By construction (see 3.4), the morphism $q_\gamma : W_\gamma \to Z$ factors through $\rho_\gamma : W_\gamma \to X_\gamma$, in such a way that $W_\gamma \mid_{X_f}$ is isomorphic to the restriction of the line bundle $p : L_\gamma = \mathbf{Spec}\left(\mathbf{S}\left(\mathcal{L}_\gamma\right)\right) \to X_\gamma$ over X_f, $f \in$ Leaves. This line bundle L_γ has a natural structure of group scheme as it represents the group functor

$$\left(Sch_{/X_\gamma}\right) \to (Grp), \left(Y \xrightarrow{f} X_\gamma\right) \mapsto H^0\left(Y, f^*\mathcal{L}_\gamma^\vee\right),$$

and W_γ comes naturally equipped with a left action $\xi_\gamma : L_\gamma \times_{X_\gamma} W_\gamma \to W_\gamma$ making $\rho_\gamma : W_\gamma \to X_\gamma$ a principal homogeneous L_γ-bundle.

4.1. Every nonzero section $s \in H^0\left(X_\gamma, \mathcal{L}_\gamma^\vee\right)$ gives rise to a group homomorphism

$$\phi_s : \mathbb{G}_{a,X_\gamma} = \mathbf{Spec}\left(\mathcal{O}_{X_\gamma}[T]\right) \to L_\gamma,$$

whence to a nontrivial $\mathbb{G}_{a,X_\gamma}$-action $\mu_s = \xi_\gamma \circ (\phi_s \times \mathrm{Id})$ on W_γ. By proposition 3.10 in [5], every nontrivial $\mathbb{G}_{a,Z}$-action on $q_\gamma : W_\gamma \to Z$ is induced by such a nonzero section s. Since

$K(X_\gamma) \simeq K(Z) \simeq \mathrm{Frac}(A)$,

$$H^0\left(X_\gamma, \mathcal{L}_\gamma^\vee\right) \simeq \left\{ g \in K(X_\gamma), \mathrm{div}(g) - \sum_{f \in \mathrm{Leaves}(\Gamma)} n_f x_f \geq 0 \right\} \cup \{0\}$$

and so, nonzero sections $s \in H^0\left(X_\gamma, \mathcal{L}_\gamma^\vee\right)$ are in one-to-one correspondence with regular functions $h^m g \in A$, where $m \geq h(\Gamma) = \max(n_f)_{f \in \mathrm{Leaves}(\Gamma)}$ and $g \in A \setminus hA$.

Remark 4.2. Let $s \in H^0\left(X_\gamma, \mathcal{L}_\gamma^\vee\right)$ be a nonzero section corresponding to a regular function $h^m g \in A$. Over X_f, the multiplication by h^{n_f} induces an isomorphism of $\mathcal{O}_{X_f}$-algebras $\tau_f : \mathcal{A}_\gamma \mid_{X_f} \xrightarrow{\sim} \mathcal{O}_{X_f}[U]$ such that the group co-action corresponding to the restriction of the $\mathbb{G}_{a,Z}$-action μ_s on $W_\gamma \mid_{X_f}$ is given by the $\mathcal{O}_{X_f}$-algebras homomorphism

$$\mathcal{O}_{X_f}[U] \to \mathcal{O}_{X_f}[U] \otimes_{\mathcal{O}_{X_f}} \mathcal{O}_{X_f}[T], U \mapsto U \otimes 1 + h^{m-n_f} g \otimes T$$

In other words, the restriction of μ_s to $W_\gamma \mid_{X_f}$ is simply a translation, twisted by $h^{m-n_f} g$.

We also recall the following useful result.

Proposition 4.3. ([5, Proposition 3.12]) *For a GDS $q_\gamma : W_\gamma \to Z$ defined by a weighted rooted tree $\gamma = (\Gamma, w)$, the following are equivalent.*
1) W_γ admits a free $\mathbb{G}_{a,Z}$-action.
2) All the leaves of Γ are at the same level.
3) The canonical sheaf ω_{W_γ} of W_γ is trivial.

Example 4.4. In [1] Bandman and Makar-Limanov proved that the canonical sheaf of the surface V of example 1.16 is not trivial by constructing certain explicit global holomorphic 2-forms on V. On the other hand, the fine-labelled tree $\mathfrak{g}$ corresponding to this surface has leaves at levels 1 and 2. Therefore, propositions 3.19 and 4.3 implies the same result. This shows in particular that there is no free $\mathbb{G}_{a,Z}$-action on V (see also example 4.5 below).

$\mathbb{G}_{a,Z}$-actions on the GDS's $q_\mathfrak{g} : V_\mathfrak{g} \to Z$.
In this subsection, we prove that every $\mathbb{G}_{a,Z}$-action on a GDS $q_\mathfrak{g} : V_\mathfrak{g} \to Z$ defined by a fine-labelled rooted tree extends to a $\mathbb{G}_{a,Z}$-action on the ambient space $\mathbb{A}_Z^{d(\Gamma)+1} = \mathrm{Spec}(A[\Gamma])$.

Example 4.5. The fine-labelled tree $\mathfrak{g}$ of example 1.16 corresponds to the Bandman and Makar-Limanov surface $V \subset \mathrm{Spec}(A[X_0, X_{e_0}, X_{e_1}])$ with equations

$$hX_{e_0} = X_0\left(X_0^2 - 1\right), \quad X_0 X_{e_1} - X_{e_0}\left(X_{e_0}^2 - 1\right), \quad hX_{e_1} = \left(X_0^2 - 1\right)\left(X_{e_0}^2 - 1\right).$$

Consider the following locally nilpotent A-derivation $\partial = \tilde{\partial}_{\mathfrak{g},2}$ of $A[\Gamma]$ given via

$$\left\{ \begin{array}{c} \partial(X_0) = h^2, \quad \partial(X_{e_0}) = h\left(3X_0^2 - 1\right) \\ \partial(X_{e_1}) = 2hX_0\left(X_{e_0}^2 - 1\right) + 2\left(X_0^2 - 1\right)\left(3X_0^2 - 1\right) X_{e_0} \end{array} \right. .$$

In [1, p. 579], Bandman and Makar-Limanov proved that ∂ induces a locally nilpotent A-derivation $\partial_{\mathfrak{g},2}$ of $B_\mathfrak{g}$. Note that corresponding $\mathbb{G}_{a,Z}$-action restricts to a free action on the complement of the irreducible components

$$C_{f_1} = \{h = 0, X_0 = 1\} \cap V \quad \text{and} \quad C_{f_2} = \{h = 0, X_0 = -1\} \cap V$$

of $q_\mathfrak{g}^{-1}(z_0)$.

More generally, the following proposition asserts that a *GDS* $q_{\mathfrak{g}} : V_{\mathfrak{g}} \to Z$ admits canonical $\mathbb{G}_{a,Z}$-actions which are the restrictions to $V_{\mathfrak{g}}$ of certain $\mathbb{G}_{a,Z}$-actions on $\mathbb{A}_Z^{d(\Gamma)+1}$.

Proposition 4.6. *For every $m \geq h(\Gamma)$, the A-derivation $\tilde{\partial}_{\mathfrak{g},m}$ of $A_h[\Gamma]$ defined recursively by*

$$\tilde{\partial}_{\mathfrak{g},m} = h^m \frac{\partial}{\partial X_0} + h^{-1} \sum_{e \in \mathfrak{P}(\Gamma)} \tilde{\partial}_{\mathfrak{g},m}(Q_e(\mathfrak{g})) \frac{\partial}{\partial X_e}$$

restricts to a triangular derivation of $A[\Gamma]$. It induces a locally nilpotent A-derivation $\partial_{\mathfrak{g},m}$ of the A-algebra $B_{\mathfrak{g}} = A[\Gamma]/I_{\mathfrak{g}}$.

Proof. Given a node $e \in \mathbf{P}(\Gamma)$ at level $k < h(\Gamma)$, $Q_e(\mathfrak{g})$ only involves the variables X_0 and $X_{e'}$, $e' \in \mathrm{Anc}(e)$ (see definition 1.9). We conclude recursively that

$$\tilde{\partial}_{\mathfrak{g},m}(X_e) = h^{-1} \sum_{e' \in \mathrm{Anc}(e) \cup \{0\}} \frac{\partial Q_e(\mathfrak{g})}{\partial X_{e'}} \tilde{\partial}_{\mathfrak{g},m}(X_{e'}) \in h^{m-k-1} A\Big[X_0, (X_{e'})_{e' \in \mathrm{Anc}(e)}\Big].$$

Thus $\tilde{\partial}_{\mathfrak{g},m}$ restricts to a triangular A-derivation of $A[\Gamma]$ as $m \geq h(\Gamma)$. By construction, $\tilde{\partial}_{\mathfrak{g},m}$ annihilates $\Delta_{0,e}(\mathfrak{g})$ for every $e \in \mathfrak{P}(\Gamma)$. Given $(e, e') \in \mathbf{P}(\Gamma) \times \mathrm{Anc}(e)$, we deduce from (1.5) that

$$h\tilde{\partial}_{\mathfrak{g},m}\left(\Delta_{e',e}(\mathfrak{g})\right) = \tilde{\partial}_{\mathfrak{g},m}\left(h\Delta_{e',e}(\mathfrak{g})\right) \in I_{\mathfrak{g}}.$$

Since $B_{\mathfrak{g}}$ is an integral A-algebra, we conclude that $\tilde{\partial}_{\mathfrak{g},m}\left(\Delta_{e',e}(\mathfrak{g})\right) \in I_{\mathfrak{g}}$ for otherwise h is a zero divisor in $B_{\mathfrak{g}}$. Thus $\tilde{\partial}_{\mathfrak{g},m}(I_{\mathfrak{g}}) \subset I_{\mathfrak{g}}$ and so, $\tilde{\partial}_{\mathfrak{g},m}$ induces a locally nilpotent A-derivation $\partial_{\mathfrak{g},m}$on $B_{\mathfrak{g}}$. □

In [6], Fieseler exploited the vector field associated to a $\mathbb{C}_+$-action to introduce the concept of fixed point order of a $\mathbb{C}_+$-action along an invariant subscheme. Here we give an equivalent definition in terms of locally nilpotent derivations.

Definition 4.7. Let $V = \mathrm{Spec}(B)$ be an integral affine scheme over a field k of characteristic 0. Given a nontrivial $\mathbb{G}_{a,k}$-action $\alpha : \mathbb{G}_{a,k} \times V \to V$ corresponding to a locally nilpotent derivation $\partial \in \mathrm{LND}_k(B)$ and an invariant subscheme $Y \subset V$ with defining ideal $I_Y \subset B$, the *fixed point order $\mu(\alpha, Y)$ of α along Y* is the maximal number $n \geq 0$ such that $\partial(B) \subset I_Y^n B$.

We have the following result.

Lemma 4.8. *For every $m \geq h(\Gamma)$ the $\mathbb{G}_{a,Z}$-action $\alpha_{\mathfrak{g},m}$ on $V_{\mathfrak{g}}$ corresponding to the derivation $\partial_{\mathfrak{g},m}$ restricts to a free action on $V_{\mathfrak{g}} \times_Z Z_*$. It has fixed point order $\mu(\alpha_{\mathfrak{g},m}, C_e) = m - n_e$ along an irreducible component $C_e \subset q_{\mathfrak{g}}^{-1}(z_0)$ corresponding to a leaf e of Γ at level n_e.*

Proof. Clearly, $\alpha_{\mathfrak{g},m}$ restricts to a free action on $V_{\mathfrak{g}} \times_Z Z_* \simeq \mathrm{Spec}(A_h[X_0])$ (see 2.4) as $\tilde{\partial}_{\mathfrak{g},m}(X_0) = h^m$. With the notation of (1.1), the ideal $\tilde{I}_{\mathfrak{g}}(e) \subset B_{\mathfrak{g}} = A[\Gamma]/I_{\mathfrak{g}}$ of the curve C_e is generated by the image of the ideal

$$I_{\mathfrak{g}}(e) = \left(I_{\mathfrak{g}}, h, X_0 - lb(e_1), \ldots, X_{e_{n_e-2}} - lb(e_{n_e})\right) \subset A[\Gamma]$$

in $B_{\mathfrak{g}}$ (see proposition 2.11). By definition of $\tilde{\partial}_{\mathfrak{g},m}$,

$$\tilde{\partial}_{\mathfrak{g},m}\left(h, X_{e_0}, \ldots, X_{e_{n_e-1}}\right) \subset I_{\mathfrak{g}}^{m-n_e}(e) \setminus I_{\mathfrak{g}}^{m-n_e+1}(e).$$

For $e' \in \mathbf{P}(\Gamma) \setminus (\downarrow e)$ there exists an indice k, $0 \leq k \leq n_e - 1$ such that e_k is the first common ancestor of e' and e. Letting $e" = \mathrm{Child}(e_k) \cap (\downarrow e')$, we have $Q_{e'} = R_{e''} Q_{e_k,e'}$, where $Q_{e_k,e'}$

only involves the variables corresponding to nodes in $[e_k, \mathrm{Par}(e')]_\Gamma$. Since $\left(X_{e_{k-1}} - lb(e_{k+1})\right)$ divides $R_{e''}$ we conclude by induction that

$$\tilde{\partial}_{\mathfrak{g},m}(X_{e'}) = h^{-1}\tilde{\partial}_{\mathfrak{g},m}(Q_{e'}) \in I_{\mathfrak{g}}^{m-k-1}(e) \subset I_{\mathfrak{g}}^{m-n_e}(e).$$

Thus $\partial_{\mathfrak{g},m}(B_{\mathfrak{g}}) \subset \tilde{I}_{\mathfrak{g}}^{m-n_e}(e) \setminus \tilde{I}_{\mathfrak{g}}^{m-n_e-1}(e)$ and so, $\mu(\alpha_{\mathfrak{g},m}, C_e) = m - n_e$. □

This leads to the following description.

Theorem 4.9. *Every nontrivial $\mathbb{G}_{a,Z}$-action on a GDS $q_{\mathfrak{g}} : V_{\mathfrak{g}} \to Z$ is induced by a locally nilpotent derivation $g\partial_{\mathfrak{g},m}$ of $B_{\mathfrak{g}}$, where $m \geq h(\Gamma)$ and $g \in A \setminus hA$.*

Proof. Since A is contained in the kernel of $\partial_{\mathfrak{g},m}$, $g\partial_{\mathfrak{g},m}$ is again locally nilpotent, whence induces a nontrivial $\mathbb{G}_{a,Z}$-action α on $V_{\mathfrak{g}}$. For every $e \in \mathrm{Leaves}(\Gamma)$, the open subset

$$V_e = \left(V_{\mathfrak{g}} \setminus q_{\mathfrak{g}}^{-1}(z_0)\right) \cup C_e$$

is Z-isomorphic to $\mathbb{A}^1_Z = \mathrm{Spec}(A[U])$ (see 3.3). By lemma 4.8, $\mu(\alpha, C_e) = m - n_e$, where $n_e = l(\downarrow e)$. We conclude similarly that $\mu\left(\alpha, q_{\mathfrak{g}}^{-1}(z)\right) = \mathrm{ord}_z(g)$ for every $z \in Z_*$. Letting $\mathbb{G}_{a,Z} = \mathrm{Spec}(A[T])$, this means that the group co-action corresponding to the restriction of α to V_f is given by the A-algebras homomorphism

$$A[U] \to A[U] \otimes_A A[T], U \mapsto U \otimes 1 + h^{m-n_e} g \otimes T.$$

By proposition 3.19, the GDS $q_\gamma : W_\gamma \to Z$ defined by the weighted rooted tree $\gamma = (\Gamma, w)$ corresponding to the pair $\left(q_{\mathfrak{g}} : V_{\mathfrak{g}} \to Z, pr_{X_0} : V_{\mathfrak{g}} \to \mathbb{A}^1_Z\right)$ is isomorphic to $V_{\mathfrak{g}}$ via the open embedding $i_\gamma : W_\gamma \hookrightarrow \mathrm{Spec}(A[\Gamma])$. We deduce from remark 4.2 that this isomorphism is equivariant when we equip $V_{\mathfrak{g}}$ with α and W_γ with the action μ_s corresponding to $h^m g \in A$ (see 4.1). Since every nontrivial $\mathbb{G}_{a,Z}$-action on W_γ is of the form μ_s for some section $s \in H^0\left(X_\gamma, \mathcal{L}_\gamma^\vee\right)$ corresponding to a regular function $h^m g \in A$, where $m \geq h(\Gamma)$ and $g \in A \setminus hA$, we conclude that every $\mathbb{G}_{a,Z}$-action on $V_{\mathfrak{g}}$ is induced by a locally nilpotent derivation $g\partial_{\mathfrak{g},m}$ for some $g \in A \setminus hA$. □

As a consequence of theorems 3.1 and 4.9, we obtain the following result.

Corollary 4.10. *Any GDS $q : V \to Z$ can be embedded into an affine space $\mathbb{A}^N_Z$ in such a way that every $\mathbb{G}_{a,Z}$-action on V extends to a $\mathbb{G}_{a,Z}$-action on $\mathbb{A}^N_Z$.*

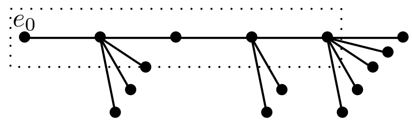
5. *GDS*'s with a trivial Makar-Limanov invariant

In this section we characterize GDS's $q_{\mathfrak{g}} : V_{\mathfrak{g}} \to Z$ with a trivial Makar-Limanov invariant. By theorem 7.7 in [5], every normal affine surface S with a trivial Makar-Limanov invariant is a cyclic quotient of a GDS V. In case that S is a log $\mathbb{Q}$-homology planes with a trivial Makar-Limanov invariant, we construct this GDS V and the cyclic group action explicitly.

Embeddings of *GDS*'s with a trivial Makar-Limanov invariant.
We recall that an (*oriented*) *comb* of height n is a rooted tree Γ such that for every node $e \in N(\Gamma) \setminus \mathrm{Leaves}(\Gamma)$ of degree $\deg_\Gamma(e) \geq 1$, all but possibly one of the children of e are leaves of Γ. This means equivalently that

$$C_\Gamma = (N(\Gamma) \setminus \mathrm{Leaves}(\Gamma)) = \{e_0 < \ldots < e_{n-1}\} \tag{5.1}$$

is either empty or a chain of length $n - 1$.



A comb rooted in e_0.

We have the following result.

Theorem 5.1. *A GDS $q_{\mathfrak{g}} : V_{\mathfrak{g}} \to Z$ has a trivial Makar-Limanov invariant if and only if $\mathfrak{g} = (\Gamma, lb)$ is a fine-labelled comb.*

Proof. By proposition 3.19, $V_{\mathfrak{g}}$ is Z-isomorphic to the GDS $q_\gamma : W_\gamma \to Z$ defined by the weighted rooted tree γ corresponding to the pair $\left(q_{\mathfrak{g}} : V_{\mathfrak{g}} \to Z, pr_{X_0} : V_{\mathfrak{g}} \to \mathbb{A}^1_Z\right)$. Since γ has the same underlying tree Γ as $\mathfrak{g}$, the statement follows from theorem 7.2 in [5] which asserts that a GDS $q_\gamma : W_\gamma \to Z$ has a trivial Makar-Limanov invariant if and only if Γ is a comb. $\square$

5.2. Given a nontrivial fine-labelled comb $\mathfrak{g} = (\Gamma, lb)$ of height n, the ideal $I_{\mathfrak{g}} \subset A[\Gamma] = A[X_0]\left[X_{e_0}, \ldots, X_{e_{n-1}}\right]$ of $V_{\mathfrak{g}}$ is generated by the polynomials

$$\left\{\begin{array}{lcll} \Delta_{0,e_i}(\mathfrak{g}) & = & hX_{e_i} - Q_{e_i}(\mathfrak{g}) & 0 \leq i \leq n-1 \\ \Delta_{e_j,e_i}(\mathfrak{g}) & = & \left(X_{e_{j-1}} - lb(e_{j+1})\right) X_{e_i} - X_{e_j} Q_{e_j,e_i}(\mathfrak{g}) & 0 \leq j < i \leq n-1, \end{array}\right. \tag{5.2}$$

These equations can be put in the following more convenient form. Letting

$$P_i(T) = (T - \lambda_{i,1}) \tilde{P}_i(T) = (T - lb(e_i)) S_{e_{i-1}}^{\{e_i\}}(\mathfrak{g})(T) \in \mathbb{C}[T], \quad 1 \leq i \leq n,$$

it is easily seen that $V_{\mathfrak{g}}$ is isomorphic to the surface

$$V_{P_1,\ldots,P_n} \subset \mathbb{A}^{n+2}_{\mathbb{C}} = \operatorname{Spec}\left(\mathbb{C}[x][y_1, \ldots, y_{n+1}]\right)$$

with equations

$$\left\{\begin{array}{lcll} xy_{i+1} & = & \left(\prod_{k=1}^{i-1} \tilde{P}_k(y_k)\right) P_i(y_i) & 1 \leq i \leq n \\ (y_{j-1} - \lambda_{j-1,1})\, y_{i+1} & = & y_j \left(\prod_{k=j}^{i-1} \tilde{P}_k(y_k)\right) P_i(y_i) & 2 \leq j \leq i \leq n \end{array}\right. ,$$

where, by convention, $\prod_{k=j}^{i-1} \tilde{P}_k(y_k) = 1$ if $i = j$. In particular, if $\mathfrak{g}$ is a fine-labelled comb of height 1 then $V_{\mathfrak{g}}$ is the ordinary Danielewski surface with equation $xy_2 - P_1(y_1) = 0$.

Remark 5.3. Note that for every collection of nonconstant polynomials $P_1, \ldots, P_n$ with simple roots, the surface $V = V_{P_1,\ldots,P_n}$ admits two natural $\mathbb{A}^1$-fibrations $pr_x : V \to \operatorname{Spec}(\mathbb{C}[x])$ and $pr_{y_{n+1}} : V \to \operatorname{Spec}(\mathbb{C}[y_{n+1}])$. By construction $pr_x : V \to \operatorname{Spec}(\mathbb{C}[x])$ is a GDS over Z. However, it may happen that $pr_{y_{n+1}} : V \to \operatorname{Spec}(\mathbb{C}[y_{n+1}])$ is a not a GDS, due to the fact that the fiber $pr_{y_{n+1}}^{-1}(0)$ is not reduced (see example 5.5 below)

Proposition 4.3 implies the following result due to Bandman and Makar-Limanov [1, Theorem 3].

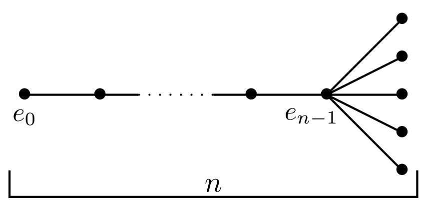
Theorem 5.4. *For a GDS $q : V \to Z$ with a trivial Makar-Limanov invariant, the following are equivalent.*

1) V admits a free $\mathbb{G}_{a,Z}$-action.

2) The canonical sheaf ω_V of V is trivial.

3) V is isomorphic to an ordinary Danielewski surface $V_{P,1} \subset \mathbb{A}^3_{\mathbb{C}} = Spec\left(\mathbb{C}[x,y,z]\right)$ with the equation $xz - P(y) = 0$ for a certain nonconstant polynomial P with simple roots.

Proof. By virtue of proposition 4.3, (1) and (2) are equivalent. Clearly, (3)⇒(2). So it remains to show that (2)⇒(3). By theorem 3.1, there exists a fine-labelled rooted tree $\mathfrak{g} = (\Gamma, lb)$ and a Z-isomorphism $\psi : V \xrightarrow{\sim} V_{\mathfrak{g}}$. By theorem 5.4, $\mathfrak{g}$ is a comb as V has a trivial Makar-Limanov invariant. Since ω_V is trivial, proposition 4.3 implies that all the leaves of Γ are at the same level $n \geq 0$. Therefore Γ has the following structure:



If $n = 0$ then $V_{\mathfrak{g}} \simeq \mathbb{A}^1_Z = \mathrm{Spec}\left(A[X_0]\right)$. Otherwise, letting $X_{e_{-1}} = X_0$, equations (5.2) simplify as

$$\left\{\begin{array}{lcll} hX_{e_i} & = & X_{e_{i-1}} - lb\left(e_{i+1}\right) & 0 \leq i \leq n-2 \\ hX_{e_{n-1}} & = & \prod_{f \in \mathrm{Child}(e_{n-1})} \left(X_{e_{n-2}} - lb\left(f\right)\right) & \end{array}\right. ,$$

and so, the statement follows. □

By [8], any two free $\mathbb{C}_+$-actions on an ordinary Danielewski surface $V \subset \mathrm{Spec}\left(\mathbb{C}(x,y,z)\right)$ with equation $xz - P(y) = 0$ are conjugated under the action of the automorphism group $\mathrm{Aut}(V)$ of V. In particular, every two $\mathbb{A}^1$-fibrations $q : V \to Z$ and $\tilde{q} : V \to \tilde{Z}$ are conjugated, in the sense that there exists an isomorphism $\bar{\theta} : Z \xrightarrow{\sim} \tilde{Z}$ and an automorphism $\theta : V \xrightarrow{\sim} V$ of V such that $\bar{\theta} \circ q = \tilde{q} \circ \theta$. In [2], Daigle and Russel give examples of nonconjugated $\mathbb{A}^1$-fibrations on a log $\mathbb{Q}$-homology plane with a trivial Makar-Limanov invariant. In case that $q : V \to Z$ is a *GDS*, the following example shows that there may also exist nonconjugated $\mathbb{A}^1$-fibrations on V.

Example 5.5. By example 1.16, the Bandman and Makar-Limanov surface $V \subset \mathrm{Spec}\left(\mathbb{C}[x,y,z,u]\right)$ with equations

$$xz = y\left(y^2 - 1\right), \quad yu = z\left(z^2 - 1\right), \quad xu = \left(y^2 - 1\right)\left(z^2 - 1\right)$$

is a *GDS* via $pr_x : V \to Z = \mathrm{Spec}\left(\mathbb{C}[x]\right)$. It is also a *GDS* via $pr_u : V \to \tilde{Z} = \mathrm{Spec}\left(\mathbb{C}[u]\right)$. The $\mathbb{C}$-algebra involution

$$\mathbb{C}[x,u][y,z] \to \mathbb{C}[x,u][y,z], x \leftrightarrow u, y \leftrightarrow z,$$

induces an automorphism $\theta : V \xrightarrow{\sim} V$. Letting $\bar{\theta} : Z \xrightarrow{\sim} \tilde{Z}$ be the isomorphism corresponding to the induced isomorphism $\mathbb{C}[u] \xrightarrow{\sim} \mathbb{C}[x]$, we conclude that $\bar{\theta} \circ pr_x = pr_u \circ \theta$.

Similarly, the surface $V' \subset \mathbb{C}[x,y,z,u]$ with equations

$$xz = y(y-1), \quad yu = z^2, \quad xu = (y-1)z$$

is a GDS $pr_x : V' \to Z = \operatorname{Spec}(\mathbb{C}[x])$. The natural second $\mathbb{A}^1$-fibration $pr_u : V' \to \tilde{Z}$ restricts to the trivial line bundle over $\tilde{Z} \setminus \{0\}$ but the fiber

$$pr_u^{-1}(0) \simeq \operatorname{Spec}\left(\mathbb{C}[x,y,z] / \left(z^2, (y-1)z, xz - y(y-1)\right)\right)$$

is not reduced. Therefore, there is no automorphism $\theta : V' \xrightarrow{\sim} V'$ satisfying $\bar{\theta} \circ pr_x = pr_u \circ \theta$.

Log $\mathbb{Q}$-homology planes with a trivial Makar-Limanov invariant.
A log $\mathbb{Q}$-homology plane is a normal affine surface S with log-terminal singularities and with vanishing singular homology groups $H_i(S, \mathbb{Q})$ for all $i > 0$. In this subsection we prove the following result (compare with [11, Theorem 3.4]).

Theorem 5.6. *Every log $\mathbb{Q}$-homology plane $S \not\simeq \mathbb{A}^2_{\mathbb{C}}$ with a trivial Makar-Limanov invariant is isomorphic to the quotient of an ordinary Danielewski surface $V \subset \operatorname{Spec}(\mathbb{C}[x,t,z])$ with equation $xz = t^n - 1$ by a $\mathbb{Z}_m$-action $(x,t,z) \mapsto \left(\varepsilon x, \varepsilon^q t, \varepsilon^{-1} z\right)$, where ε is a primitive m-th root of unity, n divides m and $\gcd(q, m/n) = 1$.*

The proof is given in 5.7-5.8 below. Since S has a trivial Makar-Limanov invariant, it admits an $\mathbb{A}^1$-fibration $\rho : S \to Y \simeq \operatorname{Spec}(\mathbb{C}[y])$ such that all but possibly one closed fiber, say $\rho^{-1}(y_0)$, where $y_0 = (y) \in Y$, are isomorphic to $\mathbb{A}^1_{\mathbb{C}}$ (see e.g. [4, Proposition 2.15]). In turn, lemma 1.2 in [6] and [5, 2.5] imply that ρ restricts to a trivial line bundle over $Y_* = Y \setminus \{y_0\}$. By Theorem 4.3.1 in [10] p.231, $\rho^{-1}(y_0) = mC$, where $m \geq 1$ and $C \simeq \mathbb{A}^1_{\mathbb{C}}$. In particular, there exists k, $0 \leq k \leq m-1$, such that the canonical sheaf ω_S of S is isomorphic to $\mathcal{O}_S(kC)$.

5.7. If $m = 1$ then $S \simeq \mathbb{A}^1_Y$ (see e.g. [10, Theorem 4.3.1 p.231]). Otherwise, if $m \geq 2$ then we let $\theta : Z \simeq \mathbb{A}^1_{\mathbb{C}} \to Y$ be a Galois covering of order m, unramified over Y_* and totally ramified over y_0. Up to an isomorphism, we can suppose that θ is induced by the $\mathbb{C}$-algebras homomorphism $\theta^* : \mathbb{C}[y] \to \mathbb{C}[x,y]/(x^m - y) \simeq \mathbb{C}[x]$, where the Galois group $\mathbb{Z}_m$ of m-th roots of unity acts on $\mathbb{C}[x,y]$ via $\varepsilon \cdot f(x,y) = f\left(\varepsilon^{-1}x, y\right)$. The group $\mathbb{Z}_m$ acts on the normalization V of the fiber product $S \times_Y Z$ and $S \simeq V/\mathbb{Z}_m$. This surface V inherits an $\mathbb{A}^1$-fibration $q : V \to Z$ with reduced fibers, induced by the projection $pr_Z : S \times_Y Z \to Z$. Letting $z_0 = (x) \in Z$, the group $\mathbb{Z}_m$ acts transitively on the set of irreducible components $C_1, \ldots, C_n$ of $q^{-1}(z_0)$, where n divides m (see e.g. [6, Theorem 1.7]). Since $\rho : S \to Y$ restricts to a trivial line bundle over Y_*, we conclude that $q : V \to Z$ restricts to a trivial line bundle over $Z_* = Z \setminus \{z_0\}$ and so, $q : V \to Z$ is a GDS over Z. By construction, the quotient morphism $\phi : V \to S$ is étale over S_{reg}, whence is étale in codimension 1 as S is normal. Thus the canonical sheaf ω_V of V is isomorphic to $\phi^*(\omega_S) \simeq q^*(\mathcal{O}_Z(kz_0))$, hence is trivial. Moreover, we deduce from [12] that the Makar-Limanov invariant of V is trivial.

5.8. By theorem 5.4, V is isomorphic to an ordinary Danielewski surface $V_{P,1} \subset \operatorname{Spec}(\mathbb{C}[x,t,z])$ with the equation $xz - P(t) = 0$ for a certain nonconstant polynomial P with n simple roots $t_1, \ldots, t_n \in \mathbb{C}$. Since $\mathbb{Z}_m$ acts transitively on the set of irreducible components of $q^{-1}(z_0)$ and t is locally constant on $q^{-1}(z_0)$ with the value t_i on C_i, $1 \leq i \leq n$, we can suppose that $t_{i+1} = \varepsilon^q t_i$ where ε is a primitive m-th root of unity and $\gcd(q, m/n) = 1$. Up to an isomorphism, we can further suppose that $t_1 = 1$ (see e.g. [8]). The canonical morphism $pr_{x,t} : V \to \mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec}(\mathbb{C}[x,t])$ expresses V as the affine modification [7] of $\mathbb{A}^2_{\mathbb{C}}$ with the locus $(I, f) = ((x, P(t)), x)$. Clearly, $pr_{x,t}$ is $\mathbb{Z}_m$-equivariant when we equip $\mathbb{A}^2_{\mathbb{C}}$ with the $\mathbb{Z}_m$-action $(x,t) \mapsto (\varepsilon x, \varepsilon^q t)$. As $\Gamma(V, \mathcal{O}_V) \simeq \mathbb{C}[x, t, P(t)/x]$, we conclude that V is isomorphic to the ordinary Danielewski surface with equation $xz = t^n - 1$, equipped with the $\mathbb{Z}_m$-action $(x,t,z) \mapsto \left(\varepsilon x, \varepsilon^q t, \varepsilon^{-1} z\right)$.

Remark 5.9. Note that $S \simeq V/\mathbb{Z}_m$ is nonsingular if and only if $n = m$. On the other hand, if $n = 1$ then S is isomorphic to a cyclic quotient of $\mathbb{A}^2 = \operatorname{Spec}(\mathbb{C}[x,t])$ by the $\mathbb{Z}_m$-action $(x,t) \mapsto (\varepsilon x, \varepsilon^q t)$, where $\gcd(q,m) = 1$. In particular, every affine toric surface nonisomorphic to $\mathbb{C}\times\mathbb{C}^*$ has a trivial Makar-Limanov invariant, see also [4, Proposition 2.11] for a geometrical proof.

Corollary 5.10. *Every log $\mathbb{Q}$-homology plane with a trivial Makar-Limanov invariant admits a nontrivial $\mathbb{C}^*$-action.*

Proof. Indeed, every ordinary Danielewski surface $V_{P,1} \subset \mathbb{A}^3_{\mathbb{C}} = \operatorname{Spec}(\mathbb{C}[x,t,z])$ with equation $xz - P(t) = 0$ admits a nontrivial $\mathbb{C}^*$-action induced by the $\mathbb{C}^*$-action on $\mathbb{A}^3_{\mathbb{C}}$ $\lambda \cdot (x,t,z) = (\lambda x, y, \lambda^{-1} z)$, $\lambda \in \mathbb{C}^*$. In case that $P(t) = t^n - 1$, this action commutes with the $\mathbb{Z}_m$-action $(x,t,z) \mapsto (\varepsilon x, \varepsilon^q t, \varepsilon^{-1} z)$ and so, the quotient $S = V/\mathbb{Z}_m$ inherits a nontrivial $\mathbb{C}^*$-action. □

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